Ellipticity and the problem of iterates in Denjoy-Carleman classes

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The Problem of Iterates: Introduction

Theorem (Kotake–Naramsimhan/Komatsu)

Let P be a differential operator with real-analytic coefficients in an open set $\Omega \subseteq \mathbb{R}^n$. Then a smooth function $f \in \mathcal{C}^\infty(\Omega)$ is real-analytic if and only if for each compact $K \subseteq \Omega$ there are constants $C, h > 0$ such that

$$\left\| P^k u \right\|_{L^2(K)} \leq C h^k k! \qquad \forall\, k \in \mathbb{N}_0.$$

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Problem of Iterates

Let P be a linear differential operator. If u is a function(distribution) such that the iterates $P^k u$ satisfy uniform estimates can we conclude that all derivatives of u satisfy these uniform estimates?

Gevrey vectors

We denote the Gevrey class of order $s \geq 1$ by $\mathcal{G}^s(\Omega)$ (Ω will always denote an open set in $\mathbb{R}^n$.) Let

$$P = \sum_{|\alpha| \leq d} p_\alpha D^\alpha \qquad D_j = -i\partial_j, \; p_\alpha \in \mathcal{G}^s(\Omega).$$

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A distribution $u \in \mathcal{D}'(\Omega)$ is an s-Gevrey vector of P if $P^k u \in L^2_{loc}(\Omega)$ for all $k \in \mathbb{N}_0$ and for each compact set $K \subseteq \Omega$ there is a constant $C > 0$ such that

$$\left\| P^k u \right\|_{L^2(K)} \leq C^{k+1} (dk)!^s, \qquad \forall\, k \in \mathbb{N}_0.$$

The space of s-Gevrey vectors of P is $\mathcal{G}^s(\Omega; P)$.

The Theorem of Iterates in Gevrey classes

Theorem (Lions–Magenes 1970, Bolley–Camus 1981)

Let $s \geq 1$. If P is an elliptic differential operator with coefficients in $\mathcal{G}^s(\Omega)$ then $\mathcal{G}^s(\Omega; P) = \mathcal{G}^s(\Omega)$.

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Theorem (Baouendi–Métivier 1982)

Let P be a hypoelliptic operator of principal type with real-analytic coefficients. Then the following statements hold:

1. $\mathcal{G}^1(\Omega; P) = \mathcal{G}^1(\Omega)$
2. *If $s > 1$ then for any $V \Subset \Omega$ there exists $s' > s$ such that for any $u \in \mathcal{G}^s(\Omega; P)$ we have that $u|_V \in \mathcal{G}^{s'}(V)$.*

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Theorem 1 (Metivier 1978)

Let P be a differential operators with coefficients in $\mathcal{C}^\omega(\Omega)$ and $s > 1$. Then the following statements are equivalent:

1. *P is elliptic.*
2. $\mathcal{G}^s(\Omega; P) = \mathcal{G}^s(\Omega)$.

Denjoy-Carleman Classes

Definition

We say that $\mathbf{M} = (M_k)_{k\geq 0}$ is a weight sequence if $M_0 = 1$ and

$$M_k^2 \leq M_{k-1}M_{k+1} \qquad \forall\, k \in \mathbb{N}.$$

A function $f \in \mathcal{C}^\infty(\Omega)$ is ultradifferentiable of class $\{\mathbf{M}\}$ if for all compact sets $K \subseteq \Omega$ there are constants $C, h > 0$ such that

$$\sup_{x\in K} |D^\alpha f(x)| \leq Ch^{|\alpha|} M_{|\alpha|} \qquad \forall\, \alpha \in \mathbb{N}_0^n.$$

$\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ denotes the space of all ultradifferentiable functions of class $\{\mathbf{M}\}$ in Ω.

Some basic conditions

Inclusion of real-analytic functions: If

$$\liminf_{k\to\infty}\left(\frac{M_k}{k!}\right)^{1/k} > 0 \qquad (1)$$

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Derivation Closedness: If **M** satisfies

$$\exists\, Q > 0: \quad M_{k+1} \leq Q^{k+1} M_k \quad \forall\, k \in \mathbb{N}_0, \qquad (2)$$

then any derivative of an element in $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ is itself contained in $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

We call a weight sequence **M** semiregular if **M** satisfies (1) and (2). If **M** is semiregular then $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ is closed under composition with real-analytic mappings (cf. [Hörmander 1990])

Non-Quasianalyticity I

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$$\mathcal{D}^{\{\mathbf{M}\}}(\Omega) = \mathcal{E}^{\{\mathbf{M}\}}(\Omega) \cap \mathcal{C}_0^\infty(\Omega) \neq \{0\}.$$
- Note that (3) implies (1).

Non-Quasianalyticity II

A weight sequence **M** is strongly non-quasianalytic if

$$\exists\, A > 0 : \quad \sum_{k=j}^{\infty} \frac{M_k}{M_{k+1}} \leq A(j+1)\frac{M_j}{M_{j+1}} \qquad \forall\, j \in \mathbb{N}_0.$$

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Theorem (Petzsche 1980)

The weight sequence $\mathbf{M}$ is strongly non-quasianalytic if and only if the associated Borel map

$$\begin{aligned} \mathfrak{b}_{\{\mathbf{M}\}} : \mathcal{E}^{\{\mathbf{M}\}}([-1,1]) &\longrightarrow \quad \Lambda_{\{\mathbf{M}\}} \\ f \qquad &\longmapsto \Big(f^{(k)}(0)\Big)_{k\in\mathbb{N}_0} \end{aligned}$$

is surjective.

Here we have set

$$\Lambda_{\{\mathbf{M}\}} := \Big\{(a_k)_k \in \mathbb{C}^{\mathbb{N}_0} : \exists\, C,h>0 : \ |a_k| \le C h^k M_k \quad \forall\, k\in\mathbb{N}_0\Big\}.$$

Denjoy-Carleman vectors

Let $\mathbf{M}$ be a weight sequence and P be a differential operator of order d with coefficients in $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

A distribution $u \in \mathcal{D}'(\Omega)$ is a $\{\mathbf{M}\}$-vector of P if $P^k u \in L^2_{loc}(\Omega)$ and for all compact $K \subseteq \Omega$ there are constants $C, h > 0$ such that

$$\left\| P^k u \right\|_{L^2(K)} \leq C h^k M_{dk} \qquad k \in \mathbb{N}_0.$$

$\mathcal{E}^{\{\mathbf{M}\}}(\Omega; P)$ is the space of all $\{\mathbf{M}\}$-vectors associated to P.

Remarks

- If P is an elliptic operator with real-analytic coefficients in Ω then $\mathcal{E}^{\{\mathbf{M}\}}(\Omega; P) = \mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ for any semiregular weight sequence $\mathbf{M}$, see [Bolley–Camus–Mattera 1979].

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- We need more restrictive conditions on $\mathbf{M}$ in the case that P has merely ultradifferentiable coefficients of class $\{\mathbf{M}\}$, see e.g. [Lions–Magenes, 1970], etc.

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- Some examples of weight sequences:
  - Let $s \geq 1$. The Gevrey sequence $\mathbf{G}^s$ given by $G_k^s = (k!)^s$ is strongly non-quasianalytic if and only if $s > 1$. $\mathbf{G}^s$ satisfies (2) for all $s \geq 1$.

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  - Let $q > 1$ and $r > 1$. The weight sequence $\mathbf{N}^{q,r}$ given by $N_k^{q,r} = q^{k^r}$ is strongly non-quasianalytic for all $q, r > 1$ but $\mathbf{N}^{q,r}$ satisfies (2) if and only if $1 < r \geq 2$.

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  - Let $\sigma > 0$. The weight sequence $\mathbf{L}^\sigma$ given by $L_k^\sigma = k!(\log(k+e))^{\sigma k}$ is not strongly non-quasianalytic for any choice of $\sigma > 0$. However $\mathbf{L}^\sigma$ is non-quasianalytic if and only if $\sigma > 1$. Finally, (1) and (2) hold for all $\sigma > 0$.

Main Theorem

Theorem A (F.–Schindl 2023)

Let P be a non-elliptic operator with real-analytic coefficients in Ω. If $\mathbf{M}$ is a strongly non-quasianalytic weight sequence then there is a smooth function $u \in \mathcal{C}^\infty(\Omega)$ such that

$$u \in \mathcal{E}^{\{\mathbf{M}\}}(\Omega; P) \setminus \mathcal{E}^{\{\mathbf{M}\}}(\Omega).$$

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Corollary

Let P be a differential operator with real-analytic coefficients in Ω and $\mathbf{M}$ be a strongly non-quasianalytic weight sequence which also satisfies (2). Then the following statements are equivalent:

1. *P is elliptic.*
2. $\mathcal{E}^{\{\mathbf{M}\}}(\Omega; P) = \mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

Main Theorem: Invariant Version

Theorem A'

Suppose that the Borel map associated to the Denjoy-Carleman structure $\mathcal{E}^{\{\mathbf{M}\}}$ *is surjective.*
Then for any non-elliptic partial differential operator P *there is an* $\{\mathbf{M}\}$*-vector of* P *which is not a function of class* $\{\mathbf{M}\}$*.*

Sketch of the proof in the Gevrey case: Prologue

Let

$$P = \sum_{|\alpha|\le d} p_\alpha(x) D^\alpha, \qquad p_\alpha \in \mathcal{C}^\infty(\Omega)\ \big(\text{or } \mathcal{C}^\omega(\Omega)\big);$$

$$p(x,\xi) = \sum_{|\alpha|\le d} p_\alpha(x)\xi^\alpha \qquad \ldots \text{ symbol of } P,$$

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If P is non-elliptic then there are $x_0 \in \Omega$ and $\xi_0 \in S^{n-1}$ such that

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Let $\delta > 0$ be such that $B(x_0, 2\delta) = \{y \in \mathbb{R}^n : |y - x_0| \leq 2\delta\} \subseteq \Omega$.

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Then we have

$$\exists\, D \geq 1 \ \forall t \geq 1 \ \forall \varepsilon \in (0,1) \ \forall x \in B(x_0, 2\delta t^{-\varepsilon}) :$$
$$|p(x, t\xi_0)| \leq D t^{d-\varepsilon}.$$

Sketch of proof in the Gevrey case: Part 1

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- Let $\psi \in \mathcal{G}^\sigma(\mathbb{R}^n) \cap \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| \leq \delta$ and $\psi(x) = 0$ for $|x| \geq 2\delta$.

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- The vector u is defined as

$$u(x) = \int_1^{\infty} \psi\left(t^{\varepsilon}(x - x_0)\right) e^{-t^{\eta}} e^{it\xi_0(x-x_0)}\, dt,$$

where $\eta = (d - \varepsilon)/(ds) < 1/s$.

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$$D_{\xi_0}^k u(x_0) = \int_1^\infty t^k e^{-t^\eta}\, dt = \frac{1}{\eta}\Gamma\left(\frac{k+1}{\eta}\right) + o(1).$$

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- Thus $u \notin \mathcal{G}^{s'}$ near x_0 for all $s' < 1/\eta$.

Sketch of proof in the Gevrey case: Part 2

- In order to estimate $P^k u$ we introduce functions $Q_k(x,t)$ such that
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for a generic constant A independent of k.
- Finally $|P^k u| \leq A^{k+1} (dk)!^s \int_1^\infty \exp(-t^\nu/2)$.

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- The problem for us: Optimal functions for Denjoy-Carleman classes have been previously constructed, but only as Fourier series.
- Our first aim: to construct optimal functions in DC classes as Fourier integrals.

Some Definitions and Notation

Let $\mathbf{M}$, $\mathbf{N}$ be weight sequences and $A > 0$.

$$
\begin{aligned}
\mathbf{M} \leq \mathbf{N} \quad &:\Longleftrightarrow & M_k &\leq N_k & \forall\, k \in \mathbb{N}_0, \\
\mathbf{M} \leq A\mathbf{N} \quad &:\Longleftrightarrow & M_k &\leq AN_k & \forall\, k \in \mathbb{N}_0, \\
\mathbf{M} \preceq \mathbf{N} \quad &:\Longleftrightarrow \quad \exists\, C, h > 0 : & M_k &\leq Ch^k N_k & \forall\, k \in \mathbb{N}_0, \\
\mathbf{M} \approx \mathbf{N} \quad &:\Longleftrightarrow & \mathbf{M} \preceq \mathbf{N} &\ \wedge\ \mathbf{N} \preceq \mathbf{M}, \\
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Some Definitions and Notation

Let $\mathbf{M}$, $\mathbf{N}$ be weight sequences and $A > 0$.

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\mathbf{M} \le \mathbf{N} \quad &:\Longleftrightarrow & M_k \le N_k \quad & \forall\, k \in \mathbb{N}_0,\\
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If $\mathbf{M} \preceq \mathbf{N}$ then $\mathcal{E}^{\{\mathbf{M}\}}(\Omega) \subseteq \mathcal{E}^{\{\mathbf{N}\}}(\Omega)$ and $\mathcal{E}^{\{\mathbf{M}\}}(\Omega; P) \subseteq \mathcal{E}^{\{\mathbf{N}\}}(\Omega; P)$.

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Furthermore for **M**, **N** and $\rho > 0$ we define two new weight sequences:

- **MN** which is given by $(MN)_k = M_k N_k$.
- $\mathbf{M}^\rho$ given by $(M^\rho)_k = M_k^\rho$.

Associated functions

Let $\mathbf{M}$ be a weight sequence. The weight function $\omega_{\mathbf{M}}$ associated to $\mathbf{M}$ is defined by

$$\omega_{\mathbf{M}}(t) = \sup_{k \in \mathbb{N}_0} \log \frac{t^k}{M_k}, \quad t > 0, \qquad \& \qquad \omega_{\mathbf{M}}(0) = 0.$$

Then $\omega_{\mathbf{M}}$ is a continuous function on $[0, \infty)$ which increases faster then $\log t^p$ for every $p \in \mathbb{N}$.

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The weight associated to $\mathbf{M}$ is the function

$$h_{\mathbf{M}}(t) = \inf_{k\in\mathbb{N}_0} M_k t^k, \quad t > 0, \qquad \& \qquad h_{\mathbf{M}}(0) = 0.$$

Clearly

$$h_{\mathbf{M}}\left(\frac{1}{t}\right) = e^{-\omega_{\mathbf{M}}(t)}, \qquad t > 0.$$

Hence $h_{\mathbf{M}}$ is a continuous function which is flat at the origin.

Ultraholomorphic functions

Let $\mathcal{R}$ be the Riemann surface of the logarithm. For $\gamma > 0$ let

$$S_\gamma = \left\{ z \in \mathcal{R} : |\arg z| \leq \frac{\gamma\pi}{2} \right\}.$$

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$$S_\gamma = \left\{ z \in \mathcal{R} : |\arg z| \leq \frac{\gamma\pi}{2} \right\}.$$

If $\mathbf{M}$ is a weight sequence then let $\mathcal{O}_{\{\mathbf{M}\}}(S_\gamma)$ be the space of holomorphic functions g on S_γ for which there are a formal series $\hat{g} = \sum_{k=0}^{\infty} a_k z^k$ and constants $C, h > 0$ such that

$$\left| g(z) - \sum_{j=0}^{k-1} a_j z^j \right| \leq C h^k M_k |z|^k, \forall\, z \in S_\gamma,\ \forall k \in \mathbb{N}.$$

We say that $\hat{g}$ is the $\{\mathbf{M}\}$-asymptotic expansion of g.

Remarks

- If $g \in \mathcal{O}_{\{\mathbf{M}\}}(S_\gamma)$ and $K \Subset S_\gamma$ is a subsector then there are constants $C, Q > 0$ such that

$$\sup_{z\in K}\left|g^{(k)}(z)\right| \le CQ^k M_k, \qquad \forall\, k \in \mathbb{N}_0.$$

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- If $g \in \mathcal{O}_{\{\mathbf{M}\}}(S_\gamma)$ then there is only one asymptotic expansion $\hat{g}$.
- The asymptotic Borel map $\mathfrak{b}_{\mathbf{M},\gamma} : \mathcal{O}_{\{\mathbf{M}\}}(S_\gamma) \to \mathbb{C}_{\{\mathbf{M}\}}[[z]]$ is given by $g \mapsto \hat{g}$.

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$$\mathbb{C}_{\{\mathbf{M}\}}[[z]] = \left\{ \sum_{j=0}^{\infty} a_j z^j \in \mathcal{C}[[z]] : \exists\, C, h > 0 |a_k| \leq C h^k M_k \quad \forall\, k \in \mathbb{N}_0 \right\}$$

An invariant

We say that a sequence $(c_k)_k$ is almost increasing if there is a constant $a > 0$ such that $c_\ell \leq a c_k$ for all $k \leq \ell$.

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For the study of the surjectivity of the asymptotic Borel map Thilliez introduced the following invariant:

Definition

For a weight sequence $\mathbf{M}$ we set

$$\gamma(\mathbf{M}) = \sup\left\{\gamma > 0 : \text{ The sequence } \frac{M_k}{M_{k-1}k^\gamma} \text{ is almost increasing}\right\}.$$

- The asymptotic Borel map is surjective on S_γ if and only if $\gamma < \gamma(\mathbf{M})$.
- $\gamma(\mathbf{G}^s) = s$ for $s \geq 1$.
- $\mathbf{M}$ is a strongly non-quasianalytic weight sequence if and only if $\gamma(\mathbf{M}) > 1$.
- $\gamma(\mathbf{M}^\rho) = \rho\gamma(\mathbf{M})$ for $\rho > 0$.

Optimal functions in the ultraholomorphic setting

Definition

A holomorphic function G on S_γ is an $\{\mathbf{M}\}$-optimal flat functions if

$$\begin{aligned} G(t) &\geq A_1 h_{\mathbf{M}}(B_1 t), \qquad && t > 0, \\ |G(z)| &\leq A_2 h_{\mathbf{M}}(B_2 t), \qquad && z \in S_\gamma, \end{aligned}$$

for constants $A_1, A_2, B_1, B_2 > 0$.

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for constants $A_1, A_2, B_1, B_2 > 0$.

Clearly $G \in \mathcal{O}_{\{\mathbf{M}\}}(S_\gamma)$ with asymptotic expansion $\hat{G} = 0$.

The main technical result

Theorem (Jiminez-Garrio–Sanz–Schindl 2022)

1. *If* $\gamma < \gamma(\mathbf{M})$ *then there exist* $\{\mathbf{M}\}$*-optimal flat functions in* S_γ*.*
2. *If* G *is an optimal* $\{\mathbf{M}\}$*-flat function then there are constants* $C_1, h_1 > 0$ *such that*

$$C_1 h_1^k M_k \leq \int_0^\infty t^k G(1/t)\, dt.$$

If $\mathbf{M}$ *satisfies additionally* (2) *then there exist* $C_2, h_2 > 0$ *such that*

$$\int_0^\infty t^k G(1/t)\, dt \leq C_2 h_2^k M_k.$$

Optimal functions in DC-classes

Let $\mathbf{M}$ a weight sequence and $G_{\mathbf{M}}$ an optimal $\{\mathbf{M}\}$-flat function (in some sector S_γ). If we choose $x_0 \in \mathbb{R}^n$ and $\xi_0 \in S^{n-1}$ and set

$$f(x) = \int_0^\infty G_{\mathbf{M}}(1/t) e^{i\xi_0 t(x-x_0)}\, dt$$

then

$$D_{\xi_0}^k f(x_0) = \int_0^\infty t^k G_{\mathbf{M}}(1/t)\, dt.$$

Thus f cannot be of class $\{\mathbf{T}\}$ near x_0 for any weight sequence $\mathbf{T} \precnapprox \mathbf{M}$.

If (2) holds for $\mathbf{M}$ then $f \in \mathcal{E}^{\{\mathbf{M}\}}(\mathbb{R}^n)$.

The construction of u in the DC-case

- Let $\mathbf{M}$ be a weight sequence and suppose that there are two weight sequences $\mathbf{L}$ and $\mathbf{N}$ such that $\mathbf{L}$ is non-quasianalytic, $\gamma(\mathbf{N}) > 0$ and $\mathbf{L} \preceq \mathbf{M} \precnapprox \mathbf{N}$.

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- For $x_0 \in \Omega$ let $\delta > 0$ be such that $B(x_0, 2\delta) \subseteq \Omega$.
- Let $\psi \in \mathcal{D}^{\{\mathbf{L}\}}(\mathbb{R}^n)$ be such that $\psi(x) = 1$ for $|x| < \delta$ and $\psi(x) = 0$ for $|x| > 2\delta$.

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- If $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $0 < \varepsilon < 1$ (to be specified later) then we set

$$u(x) = \int_1^\infty \psi\left(t^\varepsilon(x - x_0)\right) \Phi_N(t) e^{it\xi_0(x-x_0)}\, dt.$$

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$$u(x) = \int_1^\infty \psi\left(t^\varepsilon(x - x_0)\right) \Phi_N(t) e^{it\xi_0(x-x_0)}\, dt.$$
- Thence u is a $\mathcal{C}^\infty$-function which is not of class $\{\mathbf{T}\}$ near x_0 for any $\mathbf{T} \precnapprox \mathbf{N}$. In particular $u \notin \mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

The construction of u in the DC-case: Part II

Now let P be a linear differential operator of order d with coefficients in $\mathcal{E}^{\{\mathbf{L}\}}(\Omega)$ which is not elliptic at (x_0, ξ_0). Then there are functions Q_k such that

$$P^k u = \int_1^\infty Q_k(x,t)\Phi_{\mathbf{N}}(t)e^{it\xi_0(x-x_0)}\,dt.$$

There are constants $C, h > 0$ such that

$$|Q_k(x,t)| \leq Ch^k \left(t^{(d-\varepsilon)k} + t^{k\varepsilon(2d-1)} L_{dk} \right).$$

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Now set $\rho = t^{1-\varepsilon/d}$ and $R = t^{\varepsilon(2-1/d)}$. Obviously

$$t^{(d-\varepsilon)k} = \rho^{dk} = \rho^{dk}\frac{M_{dk}}{M_{dk}} \leq M_{dk}e^{\omega_{\mathbf{M}}(\rho)} = M_{dk}e^{\omega_{\mathbf{M}}(t^{1-\varepsilon/d})}.$$

For the second term, we need to assume that there is a weight sequence $\mathbf{V}$ such that $\mathbf{LV} \preceq \mathbf{M}$.

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If we assume that $\varepsilon \leq 1/2$ then $t^{\varepsilon(2-1/d)} \leq t^{1-\varepsilon/d}$. Therefore

$$\omega_{\mathbf{V}} \left(t^{\varepsilon(2-1/d)} \right) \leq \omega_{\mathbf{V}} \left(1 - \varepsilon/d \right).$$

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On the other hand, if we suppose that $\mathbf{V} \leq \mathbf{M}$ then $\omega_{\mathbf{M}}(s) \leq \omega_{\mathbf{V}}(s)$ for all $s \geq 0$.

Final estimates

It follows that there are constants $C, h, B_2 \geq 1$ such that

$$\left|P^k u(x)\right| \leq C h^k M_{dk} \int_1^\infty e^{-\omega_{\mathbf{N}}(t/B_2)} e^{\omega_{\mathbf{V}}(t^{1-\varepsilon/d})}\, dt$$

for $x \in \Omega$. Thence $u \in \mathcal{E}^{\{\mathbf{M}\}}(\Omega; P)$ if we can show that the integral on the right-hand side converges.

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Thus we need a way to compare $\omega_{\mathbf{N}}$ with $\omega_{\mathbf{V}}$. If we have, for example, that

$$\exists\, a \in (0,1)\ \forall\, s \geq 1 : \omega_{\mathbf{V}}(s) \leq a\omega_{\mathbf{N}}\left(B_2^{-1} s^{\frac{d}{d-\varepsilon}}\right) + D.$$

Thus

$$-\omega_{\mathbf{N}}\left(\frac{t}{B_2}\right) + \omega_{\mathbf{V}}\left(t^{1-\varepsilon/d}\right) \leq -(1-a)\omega_{\mathbf{N}}\left(\frac{t}{B_2}\right).$$

Auxillary result

Proposition (F.-Schindl 2023)

Let $\mathbf{T}$ *and* $\mathbf{U}$ *be two weight sequences and* $\tau > 1$. *Then the following two assertions are equivalent:*

1. *There is a constant* $A \geq 1$ *such that* $\mathbf{U} \leq A\mathbf{T}^{\tau}$.
2. *There is a constant* $C \geq 1$ *such that*

$$\omega_{\mathbf{T}}(s) \leq \tau^{-1}\omega_{\mathbf{U}}(s^{\tau}) + C, \qquad \forall\, s \geq 0.$$

If one of the assertions hold then for all $0 < a < 1$ *and* $\sigma \geq \tau$ *there exists a constant* $\tilde{C} \geq 1$ *such that*

$$\omega_{\mathbf{T}}(s) \leq \tau^{-1}\omega_{\mathbf{U}}(as^{\sigma}) + \tilde{C}, \qquad \forall\, s \geq 0.$$

We set $\mathbf{T} = \mathbf{V}$, $\mathbf{U} = \mathbf{N}$, $\tau = d/(d-\varepsilon)$, $a = B_2^{-1}$.

An abstract theorem

Theorem B

Let $\mathbf{M}$, $\mathbf{L}$, $\mathbf{N}$ *and* $\mathbf{V}$ *be weight sequences and* $d \in \mathbb{N}$ *such that the following properties hold:*

1. $\mathbf{M} \precnapprox \mathbf{N}$ *and* $\gamma(\mathbf{N}) > 0$
2. $\mathbf{L}$ *is non-quasianalytic.*
3. $\mathbf{V} \leq \mathbf{M}$ *and* $\mathbf{LV} \preceq \mathbf{M}$*.*
4. *There are constants* $1 < \tau < 2d/(2d-1)$ *and* $A \geq 1$ *such that* $\mathbf{N} \leq A\mathbf{V}^{\tau}$*.*

Then, for every non-elliptic differential operator P *of order* d *with coefficients in* $\mathcal{E}^{\{\mathbf{L}\}}(\Omega)$*, there is a smooth function* u *such that* $u \in \mathcal{E}^{\{\mathbf{M}\}}(\Omega; P)$ *but* $u \notin \mathcal{E}^{\{\mathbf{T}\}}(\Omega)$ *for any weight sequence* $\mathbf{T} \precnapprox \mathbf{N}$*.*

Proof of Main Theorem: Conclusion

Corollary

Let $\mathbf{M}$ *be a weight sequence with* $\gamma(\mathbf{M}) = \infty$ *and* T *be a weight sequence such that* $\mathbf{T} \preceq \mathbf{M}^{\rho}$ *for all* $\rho > 0$.
If P *is a non-elliptic differential operator with coefficients in* $\mathcal{E}^{\{\mathbf{T}\}}(\Omega)$ *then there is a function* $u \in \mathcal{C}^{\infty}(\Omega)$ *such that* $u \in \mathcal{E}^{\{\mathbf{M}\}}(\Omega; P) \setminus \mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

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Proof.

Choose parameters $0 < \sigma < 1$ and $\rho > 1$ such that

$$1 < \rho < \frac{2d}{2d-1}\sigma.$$

We set $\mathbf{V} = \mathbf{M}^\sigma$, $\mathbf{L} = \mathbf{M}^{1-\sigma}$ and $\mathbf{N} = \mathbf{M}^\rho$. Then the assumptions of Theorem C are fulfilled. □

The case $1 < \gamma(\mathbf{M}) < \infty$

The previous proof does not work in the case $1 < \gamma(\mathbf{M}) < \infty$.
But we can directly imitate the proof in the Gevrey case:
Set $\mathbf{T} = \mathbf{M}^{1/\gamma}$ with $\gamma = \gamma(\mathbf{M})$. Thus $\gamma(\mathbf{T}) = 1$ and $\gamma(\mathbf{T}^s) = s$.

Theorem C

Let $\mathbf{M}$ *be a weight sequence such that* $1 < \gamma(\mathbf{M}) < \infty$. *If* P *is a non-elliptic differential operator of class* $\{\mathbf{M}^\rho\}$, *where* $1 < 1/\rho < \gamma(\mathbf{M})$, *then there is a smooth function* u *such that*

$$u \in \mathcal{E}^{\{\mathbf{M}\}}(\Omega; P) \setminus \mathcal{E}^{\{\mathbf{M}\}}(\Omega).$$

Returning to the Gevrey case

Corollary

Let $1 \leq r < s$ *and* P *be a non-elliptic differential operator with coefficients in* $\mathcal{G}^r(\Omega)$. *Then there is a smooth function* u *such that*

$$u \in \mathcal{G}^s(\Omega; P) \setminus \mathcal{G}^s(\Omega).$$

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Let $1 \leq r < s$ and P be a non-elliptic differential operator with coefficients in $\mathcal{G}^r(\Omega)$. Then there is a smooth function u such that

$$u \in \mathcal{G}^s(\Omega; P) \setminus \mathcal{G}^s(\Omega).$$

Theorem D

Let $1 \leq r < s$. If P is a differential operator with coefficients in $\mathcal{G}^r(\Omega)$ then the following statements are equivalent:

1. *P is elliptic.*
2. $\mathcal{G}^s(\Omega; P) = \mathcal{G}^s(\Omega)$.

Other weights

Definition

A weight function is an increasing continuous function $\omega : [0,\infty) \to [0,\infty)$ with the following properties:

- $\omega|_{[0,1]} = 0$
- $\omega(2t) = O(\omega(t))$, $\to \infty$,
- $\log t = O(\omega(t))$
- $\varphi_\omega = \omega \circ \exp$ is convex.

The conjugate function of φ_ω is

$$\varphi_\omega^*(t) = \sup_{s \geq 0}(st - \varphi(s)).$$

Classes given by weight functions

A function $f \in \mathcal{C}^\infty(\Omega)$ is ultradifferentiable of class $\{\omega\}$ if for any compact $K \subseteq \Omega$ there are constants $C, h > 0$ such that

$$\sup_{x \in K} |D^\alpha u(x)| \leq C e^{1/h \varphi^*(h|\alpha|)}, \qquad \forall\, \alpha \in \mathbb{N}_0^n.$$

The space of ultradifferentiable functions of class $\{\omega\}$ is $\mathcal{E}^{\{\omega\}}(\Omega)$.

A distribution $u \in \mathcal{D}'(\Omega)$ is an $\{\omega\}$-vector of a differential operator P (with $\mathcal{E}^{\{\omega\}}(\Omega)$) if $P^k u \in L^2_{loc}(\Omega)$, $\forall k \in \mathbb{N}_0$, and for every compact set $K \subseteq \Omega$ there are constants $C, h > 0$ such that

$$\|P^k u\|_{L^2(K)} \leq C e^{\frac{1}{h}\varphi^*_\omega(hdk)}, \qquad \forall\, k \in \mathbb{N}_0.$$

The space of $\{\omega\}$-vectors of P is $\mathcal{E}^{\{\omega\}}(\Omega; P)$.

Remarks

- Let $s \geq 1$. The weight function $\omega_s(t) = \max\{0, t^s - 1\}$ generates the Gevrey class of order s, i.e. $\mathcal{E}^{\{\omega_s\}}(\Omega) = \mathcal{G}^s(\Omega)$.

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- $\mathcal{E}^{\{\omega\}}(\Omega) \cap \mathcal{C}_0^\infty(\Omega) \neq \{0\}$ if

$$\int_1^\infty \frac{\omega(t)}{t^2}\, dt < \infty. \tag{5}$$

Problem of Iterates in BMT-Classes

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Theorem (F.–Schindl 2022)

Let ω be a weight function. If P is an elliptic operator with analytic coefficients in Ω then

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Comparison I

Theorem 2 (Juan-Huguet 2010)

Let P be a differential operator with constant coefficients and ω a non-quasianalytic weight function, i.e. it satisfies (5)*. If also* (4) *holds then the following statements are equivalent:*

- *P is elliptic.*
- $\mathcal{E}^{\{\omega\}}(\Omega; P) = \mathcal{E}^{\{\omega\}}(\Omega)$.

Theorem 3 (F.–Schindl 2022)

Let P be a analytic-hypoelliptic differential operator of principal type in Ω and ω be a weight function satisfying

$$\exists\, H > 0: \quad \omega(t^2) = O(\omega(Ht)), \quad t \to \infty. \tag{6}$$

Then $\mathcal{E}^{\{\omega\}}(\Omega; P) = \mathcal{E}^{\{\omega\}}(\Omega)$.

Comparison II

- Theorem 2 is in some way a complement to Theorem A:
  - Remember if ω is a weight function which satisfies (4) then there is a weight sequence $\mathbf{M}$ such that $\mathcal{E}^{\{\omega\}} = \mathcal{E}^{\{\mathbf{M}\}}$. Moreover, there is a $s > 1$ such that $\mathcal{E}^{\{\omega\}} \subseteq \mathcal{G}^s$.

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  - Moreover, if ω satisfies (6) then

$$\bigcup_{s>1} \mathcal{G}^s(\Omega) \subseteq \mathcal{E}^{\{\omega\}}(\Omega).$$

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THANK YOU!