The Cauchy problem for p-evolution equations in Gevrey spaces

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The Cauchy problem

We consider the following Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) \qquad (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = g(x) \qquad x \in \mathbb{R} \end{cases}, \tag{1}$$

where

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_j(t,x)D_x^{p-j}, \tag{2}$$

where $D = -i\partial$, $a_{p-j} \in C([0,T], \mathcal{B}^\infty(\mathbb{R})), j = 1, \ldots, p$, and a_p is independent of x and real valued.

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State of the art

References: Agliardi, Arias Junior, Ascanelli, Boiti, Cicognani, Colombini, Baba, Ichinose, Kajitani, Reissig, Zanghirati,...

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When the coefficients a_j are complex valued some control on the behavior for $|x| \to \infty$ of the coefficients $a_j, j = 0, 1, \dots, p-1$ is required in order to obtain well-posedness at least in $H^\infty = \cap_{m \in \mathbb{R}} H^m$. or in some Gevrey classes, in general with some loss of derivatives.

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Gevrey well posedness: Only results for $p = 2$ (Kajitani-Baba (1995), Cicognani and Reissig (2014), Dreher (2003)).

Sufficient conditions: the cases $p = 2, 3$

For $P = D_t + a_2(t)D_x^2 + a_1(t,x)D_x + a_0(t,x)$. Assume

$$\mathrm{Im}\, a_1(t,x) = \mathcal{O}(|x|^{-\sigma}), \qquad |x| \to \infty$$

for some $\sigma > 0$. Then:

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- If $\sigma \in (0,1)$ and a_0, a_1 are Gevrey regular of order s_o for some $s_o \in (1, 1/(1-\sigma))$ then the problem is well posed in some Gevrey class of order θ with $\theta \in [s_o, 1/(1-\sigma))$.

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For $P = D_t + a_3(t)D_x^3 + a_2(t,x)D_x^2 + a_1(t,x)D_x + a_0(t,x)$, H^∞ well posedness assuming that

$$|\mathrm{Im} a_2(t,x)| \leq C a_3(t)\langle x\rangle^{-1}$$

$$|\mathrm{Im} a_1(t,x)| + |\mathrm{Re}\partial_x a_2)(t,x)| \leq C a_3(t)\langle x\rangle^{-\frac{1}{2}}.$$

Gevrey well-posedness for 3-evolution equations

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- The analysis of general p-evolution operators generally relies on an iterative procedure but the argument in the Gevrey case is quite technical, based on the use of pseudodifferential operators of infinite order. We decided to test it first on the case $p = 3$ and we are confident it may be applied with more iterations for higher order operators.

Gevrey-Sobolev spaces

For every $\theta \geq 1, m, \rho \in \mathbb{R}$ we denote by

$$H^m_{\rho;\theta}(\mathbb{R}) = \{u \in \mathscr{S}'(\mathbb{R}) : \langle D \rangle^m e^{\rho \langle D \rangle^{\frac{1}{\theta}}} u \in L^2(\mathbb{R})\},$$

where $e^{\rho \langle D \rangle^{\frac{1}{\theta}}}$ is the Fourier multiplier with symbol $e^{\rho \langle \xi \rangle^{\frac{1}{\theta}}}$.

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$$H^\infty_\theta(\mathbb{R}) = \bigcap_{\rho > 0} H^m_{\rho,\theta}(\mathbb{R}), \qquad \mathcal{H}^\infty_\theta(\mathbb{R}) = \bigcup_{\rho > 0} H^m_{\rho,\theta}(\mathbb{R}).$$

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Denote by $G^\theta(\mathbb{R})$ (resp. $\gamma^\theta(\mathbb{R})$) the space of all smooth functions on $\mathbb{R}$ satisfying the following condition:

$$\sup_{\alpha\in\mathbb{N}} h^{-|\alpha|}\alpha!^{-\theta} \sup_{x\in\mathbb{R}} |\partial^\alpha f(x)| < \infty$$

for some (resp. for every $h > 0$) and by $G^\theta_0(\mathbb{R})$ and $\gamma^\theta_0(\mathbb{R})$ the subspaces of compactly supported functions in $G^\theta(\mathbb{R})$ and $\gamma^\theta(\mathbb{R})$.

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$$\gamma^\theta_0(\mathbb{R}) \subset H^\infty_\theta(\mathbb{R}) \subset \gamma^\theta(\mathbb{R}),$$
$$G^\theta_0(\mathbb{R}) \subset \mathcal{H}^\infty_\theta(\mathbb{R}) \subset G^\theta(\mathbb{R}).$$

Theorem 1 (Ann. Scuola Norm. Sup. Pisa (2024?)) Let $s_0 > 1$ and $\sigma \in (\frac{1}{2}, 1)$ such that $s_0 < \frac{1}{2(1-\sigma)}$. Let moreover $P(t, x, D_t, D_x)$ defined by (2). Assume that $a_j \in C([0, T]; \mathcal{B}^\infty(\mathbb{R}))$ for $j = 0, 1, 2$ and that

(i) $a_3 \in C([0, T]; \mathbb{R})$ and there exists $C_{a_3} > 0$ such that

$$|a_3(t)| \geq C_{a_3}, \quad t \in [0, T];$$

(ii) there exists $C_{a_2} > 0$ such that the coefficient $a_2(t, x)$ satisfy $\forall \beta \in \mathbb{N}_0$:

$$|\partial_x^\beta a_2(t, x)| \leq C_{a_2}^{\beta+1} \beta!^{s_0} \langle x \rangle^{-\sigma}, \quad t \in [0, T], \, x \in \mathbb{R};$$

(iii) there exists C_{a_1} such that

$$|Im\, a_1(t, x)| \leq C_{a_1} \langle x \rangle^{-\frac{\sigma}{2}}, \quad t \in [0, T], \, x \in \mathbb{R}.$$

Then for every $\theta \in (s_0, \frac{1}{2(1-\sigma)})$ and $\forall f \in C([0, T], H^m_{\rho;\theta}(\mathbb{R})), g \in H^m_{\rho;\theta}(\mathbb{R})$ there is a unique solution $u \in C^1([0, T], H^m_{\rho';\theta}(\mathbb{R}))$ for some $\rho' \in (0, \rho)$ such that

$$\|u(t, \cdot)\|_{H^m_{\rho';\theta}} \leq C \left(\|g\|_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau, \cdot)\|_{H^m_{\rho;\theta}} \right).$$

In particular, the Cauchy problem (1) is well posed in $\mathcal{H}^\infty_\theta(\mathbb{R}) = \bigcup_{\rho > 0} H^m_{\rho;\theta}(\mathbb{R})$.

Idea of the proof

As standard in this type of problems, we cannot obtain an energy estimate via a simple application of Gronwall inequality. Namely, if we set

$$iP = \partial_t + ia_3(t)D_x^p + \sum_{j=0}^{2} ia_j(t,x)D_x^j$$

since $a_3(t)\xi^3$ is real valued, we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{L^2} &= 2Re\,(\partial_t u(t), u(t))_{L^2} \\ &= 2Re\,(iPu(t), u(t))_{L^2} - 2Re\,(ia_3(t)D_x^3 u(t), u(t))_{L^2} \\ &\quad -2\sum_{j=0}^{2} Re\,(a_j(t,x)D_x^j)u(t), u(t))_{L^2}.\end{aligned}$$

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However, the red term contains an operator of order 2, which cannot be neglected so we cannot derive an energy inequality in L^2 from the estimate above. The idea is then to conjugate the operator iP by an operator $Q(t,x,D)$ in order to obtain $i\widetilde{P} = Q(iP)Q^{-1} = \partial_t + ia_3(t)D_x^3 + \widetilde{a}_2(t,x,D) + \widetilde{a}_1(t,x,D)$, where $\widetilde{a}_j, j = 1,2$ have order j but with $Re\,\widetilde{a}_j \geq 0$.

Idea of the proof

The operator Q used in the proof turns out to be an operator with symbol

$$Q_{\Lambda,k,\rho'}(t,x,\xi) = e^{\rho'\langle D\rangle_h^{1/\theta} + k(T-t)\langle D\rangle_h^{2(1-\sigma)}} \circ e^{\Lambda}(x,D)$$

with $\Lambda = \lambda_2 + \lambda_1 \in S_\mu^{2(1-\sigma)}$ for some $\mu > 1$, $k > 0$, $\rho' \in (0,\rho)$ and $h \geq 1$ large enough. Here, for $m \in \mathbb{R}, \mu > 1$, we denote by $S_\mu^m(\mathbb{R}^2)$ the space of all functions $a \in C^\infty(\mathbb{R}^2)$ for which there exists a constant $C > 0$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{(x,\xi)\in\mathbb{R}^2} C^{-|\alpha|-|\beta|}(\alpha!\beta!)^{-\mu}\langle\xi\rangle^{-m+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| < +\infty.$$

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The role of each part of the transformation $Q_{\tilde{\Lambda},k,\rho'}(t,x,D)$ will be, roughly speaking, the following:

- In the transformation $e^{\Lambda}(x,D)$ the functions λ_1 and λ_2 will play two different roles: namely λ_2 will not change $a_3 D_x^3$, but it will change the operator $a_2 D_x^2$ into the sum of a positive operator plus a remainder of order 1 which satisfies the same assumptions as $a_1 D_x$, plus an error of order $2(1-\sigma)$ whereas λ_1 will not change the terms of order 2 and 3, but it will turn the terms of order 1 into the sum of a positive operator, plus a remainder of order zero, plus an error of order at least $2(1-\sigma)$;

Idea of the proof

- the transformation with $e^{k(T-t)\langle D\rangle_h^{2(1-\sigma)}}$ will not change the terms of order 1, 2 and 3, but it will correct the error of order $2(1-\sigma)$, changing it into the sum of a positive operator plus a remainder of order zero;
- Finally, the transformations with $e^{\rho'\langle D\rangle_h^{\frac{1}{\theta}}}$ simply moves the setting of the Cauchy problem from Gevrey-Sobolev spaces spaces to standard Sobolev spaces: since $2(1-\sigma) < 1/\theta$ the leading part of $Q_{\tilde{\Lambda},k,\rho'}(t,x,\xi)$ is $e^{\rho'\langle \xi\rangle_h^{\frac{1}{\theta}}}$, then the inverse of $Q_{\tilde{\Lambda},k,\rho'}(t,x,D)$ possess regularizing properties with respect to the spaces $H^m_{\rho;\theta}$, because $\rho' > 0$.

Remark. The change of variable is then expressed in terms of an operator of infinite order.

The functions $\lambda_j, j = 1, 2$ involved in the proof have the following form

$$\lambda_2(x,\xi) = M_2 w\left(\frac{\xi}{h}\right)\int_0^x \langle y\rangle^{-\sigma}\psi\left(\frac{\langle y\rangle}{\langle \xi\rangle_h^2}\right)dy, \quad (x,\xi)\in\mathbb{R}^2, \tag{3}$$

$$\lambda_1(x,\xi) = M_1 w\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{-1}\int_0^x \langle y\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle y\rangle}{\langle \xi\rangle_h^2}\right)dy, \quad (x,\xi)\in\mathbb{R}^2, \tag{4}$$

where M_1, M_2 are constants to be chosen suitably large and

$$w(\xi) = \begin{cases} 0, & |\xi|\leq 1 \\ -\mathrm{sgn}\, a_3, & |\xi| > R_{a_3} \end{cases}, \quad \psi(y) = \begin{cases} 1, & |y|\leq \frac{1}{2}, \\ 0, & |y|\geq 1, \end{cases}$$

$|\partial^\alpha w(\xi)| \leq C_w^{\alpha+1}\alpha!^\mu$, $|\partial^\beta\psi(y)|\leq C_\psi^{\beta+1}\beta!^\mu$. Starting from data $g\in H^m_{\rho;\theta}$ and $f\in C([0,T], H^m_{\rho;\theta})$ for some $\rho>0$ we consider the auxiliary Cauchy problem

$$\begin{cases} \tilde{P}_{\Lambda,k,\rho'} v = f_{\Lambda,k,\rho'} \\ v(0,x) = g_{\Lambda,k,\rho'} \end{cases} \tag{5}$$

where

$$\tilde{P}_{\Lambda,k,\rho'} = Q_{\Lambda,k,\rho'}(t,x,D)P(Q_{\Lambda,k,\rho'}(t,x,D))^{-1}$$

and

$$f_{\Lambda,k,\rho'} = Q_{\Lambda,k,\rho'}(t,x,D)f \in C([0,T],H^m), \quad g_{\Lambda,k,\rho'} = Q_{\Lambda,k,\rho'}(t,x,D)g\in H^m.$$

We have that

$$iP_{\Lambda,k,\rho'} = \partial_t + ia_3(t)D_x^3 + b_2(t,x,D_x) + b_1(t,x,D_x) + b_{2(1-\sigma)}(t,x,D_x) + b_0(t,x,D_x),$$

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We can choose the constants M_1, M_2, k appearing in $Q_{\Lambda,k,\rho'}$ such that

$$Re b_j(t,x,\xi) \geq 0, j = 1,2, \qquad b_{2(1-\sigma)}(t,x,\xi) \geq 0.$$

Then, applying sharp Gårding inequality to b_1 and $b_{2(1-\sigma)}$ and Fefferman-Phong inequality to b_2 we obtain that $b_\Lambda(t,x,D)$ is a positive operator and that the Cauchy problem (5) is well posed in $H^m(\mathbb{R})$ for every $m \in \mathbb{R}$.

We have that

$$iP_{\Lambda,k,\rho'} = \partial_t + ia_3(t)D_x^3 + b_2(t,x,D_x) + b_1(t,x,D_x) + b_{2(1-\sigma)}(t,x,D_x) + b_0(t,x,D_x),$$

where $b_j \in C([0,T], S^j(\mathbb{R}^2)), j = 0,1,2$ and $b_{2(1-\sigma)} \in C([0,T], S^{2(1-\sigma)}(\mathbb{R}^2))$.

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$$\|v(t,\cdot)\|^2_{H^m} \leq C\left(\|g\|^2_{H^m} + \int_0^t \|f(\tau,\cdot)\|^2_{H^m}\right).$$

Finally, taking $u = Q_{\Lambda,k,\rho'}(t,x,D)\}^{-1}v$, we obtain that this is a solution of the original Cauchy problem and it belongs to $C^1([0,T], H^m_{\rho-\delta}(\mathbb{R}))$ for every $\delta > 0$. The uniqueness and the energy estimate follow by standard arguments.

Sharpness of decay assumptions

In the statement of Theorem 1 we assume that $a_2 \sim \langle x \rangle^{-\sigma}$ and $\mathrm{Im}\, a_1 \sim \langle x \rangle^{-\sigma/2}$ with $\sigma \in (1/2, 1)$ and we obtain well posedness in $\mathcal{H}^{\infty}_{\theta}(\mathbb{R})$ for $\theta \in \left[\theta_0, \frac{1}{2(1-\sigma)}\right)$.

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Theorem. Assume that there exist $C, R > 0$ and $\sigma \in (0, 1)$ such that $a(x) \geq C_{\sigma} \langle x \rangle^{-\sigma}$ for $x > R$ (or $x < -R$) and $|\partial_x^{\beta} a(x)| \leq C^{\beta+1} \beta! \langle x \rangle^{-\beta}\ \forall x \in \mathbb{R}$ and $\beta \in \mathbb{N}$. If the Cauchy problem for (6) is well-posed in $H^{\infty}_{\theta}(\mathbb{R})$ for some $\theta > 1$, Then $(p-1)(1-\sigma) \leq \frac{1}{\theta}$.

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Notice that for $p = 3$ the condition $2(1-\sigma) \leq \frac{1}{\theta}$ implies in turn that $2(1-\sigma) < 1$ that is $\sigma > \frac{1}{2}$, hence for $\sigma \in \left(0, \frac{1}{2}\right]$ we cannot expect well-posedness.

Sharpness of decay assumptions

In the general case the necessary condition is the following:

$$\mathcal{H}^\infty_\theta \textit{ well-posedness } \Rightarrow (p-1)(1-\sigma) \le \frac{1}{\theta}.$$

Note: if $\sigma \in (0, \frac{p-2}{p-1}]$, then $(p-1)(1-\sigma) \ge 1 > \frac{1}{\theta}$ for all $\theta > 1$
$\rightsquigarrow$ no hope for $\mathcal{H}^\infty_\theta(\mathbb{R})$ well-posedness when $\sigma \le \frac{p-2}{p-1}$ (too slow decay)

- $a(x) = \langle x \rangle^{-\sigma}$ fulfills the assumptions.
  the Cauchy problem can be $\mathcal{H}^\infty_\theta$ well-posed only for σ large enough ($\sigma > \frac{p-2}{p-1}$).

- If $\lim_{|x|\to\infty} |a(x)|\langle x \rangle^\sigma = +\infty$ for every $\sigma \in (0,1)$, e.g. $a(x) = (\ln\langle x \rangle)^{-1}$.
  $\forall \theta > 1$ we can choose σ so small that $(p-1)(1-\sigma) > \frac{1}{\theta}$
  $\rightsquigarrow$ for every $\theta > 1$, the Cauchy problem is not $\mathcal{H}^\infty_\theta$ well-posed.

Main ideas for the proof

We show that if the Cauchy problem is well-posed in $\mathcal{H}^{\infty}_{\theta}(\mathbb{R})$, then assuming $\frac{1}{\theta} < (p-1)(1-\sigma)$ leads to a contradiction.

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We say that the Cauchy problem for the operator L_p in (6) is well-posed in $\mathcal{H}^\infty_\theta(\mathbb{R})$ if for any given $\rho_0 > 0$ and $g \in H^0_{\rho_0;\theta}(\mathbb{R})$ there exists a unique solution $u \in C^1([0,T]; H^0_{\rho;\theta}(\mathbb{R}))$ with $0 < \rho \leq \rho_0$ satisfying the energy inequality

$$\|u(t,\cdot)\|_{H^0_{\rho;\theta}(\mathbb{R})} \leq C_{\rho_0,T}\|g\|_{H^0_{\rho_0;\theta}(\mathbb{R})}, \quad \forall t \in [0,T].$$

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- Define $g_k(x) = g(x - 4\nu_k^{p-1})$, $k \in \mathbb{N}_0$, where $\nu_k \to \infty$ sequence in $\mathbb{R}^+$.

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- $g_k \in H^0_{\rho_0;\theta}(\mathbb{R})$ and (CP) $\mathcal{H}^\infty_\theta$ well-posed $\Rightarrow \exists! u_k \in C^1([0,T]; H^0_{\rho;\theta}(\mathbb{R}))$ $(\rho \le \rho_0)$ solution, and $\|u_k(t,\cdot)\|_{H^0_{\rho;\theta}(\mathbb{R})} \le C_{\rho_0,T}\|g_k\|_{H^0_{\rho_0;\theta}(\mathbb{R})}, \forall t \in [0,T]$.

Main ideas for the proof

- Localization of the solutions, so that $(x, \xi) \sim (\nu_k^{p-1}, \nu_k)$: given $h \in G_0^{\theta_h}(\mathbb{R})$ a Gevrey cutoff function ($\theta_h > 1$ close to 1, $h(x) = 1$ for $|x| \leq \frac{1}{2}$, $h(x) = 0$ for $|x| \geq 1$), define

$$v_k^{(\alpha\beta)}(t,x) := w_k^{(\alpha\beta)}(x,D)u_k(t,x)$$

$$w_k^{(\alpha\beta)}(x,\xi) := h^{(\alpha)}\left(\frac{x - 4\nu_k^{p-1}}{\nu_k^{p-1}}\right) h^{(\beta)}\left(\frac{\xi - \nu_k}{\frac{1}{4}\nu_k}\right),$$

On $\mathrm{supp} w_k^{(0,0)}$: $x \in \left[3\nu_k^{p-1}, 5\nu_k^{p-1}\right]$, $\xi \in \left[\frac{3\nu_k}{4}, \frac{5\nu_k}{4}\right]$

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- Given $\lambda \in (0,1)$ to be chosen later, $\theta_1 > \theta_h$, define $N_k := \lfloor \nu_k^{\frac{\lambda}{\theta_1}} \rfloor$ and

$$E_k(t) = \sum_{\alpha \le N_k, \beta \le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{(\alpha\beta)}(t,x)\|_{L^2(\mathbb{R})} = \sum_{\alpha \le N_k, \beta \le N_k} E_{k,\alpha,\beta}(t)$$

- By the $\mathcal{H}_\theta^\infty$ well-posedness and $\theta_1 > \theta_h$, $\forall\, t \in [0,T], k \in \mathbb{N}_0$ we have:

$$E_k(t) \le C_{T,\rho_0} \sum_{\alpha \le N_k, \beta \le N_k} C^{\alpha+\beta} (\alpha!\beta!)^{\theta_h - \theta_1} \|g_k\|_{H^0_{\rho_0,\theta}(\mathbb{R})} \le C_1 C_{T,\rho_0,g}$$

Main ideas for the proof

- By condition $\frac{1}{\theta} < (p-1)(1-\sigma)$ we are going to find (using the energy method)

$$E_k(t) \geq C_3 e^{\frac{\tilde{c}_0}{2}\nu_k^{(p-1)(1-\sigma)}} \longrightarrow_{k\to\infty} +\infty$$

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Since $\partial_t v_k^{(\alpha\beta)} = -iD_x^p v_k^{(\alpha\beta)} + a(x)D_x^{p-1}v_k^{(\alpha\beta)} + f_k^{(\alpha\beta)}$, with $f_k^{(\alpha\beta)} = [P, w_k^{(\alpha\beta)}]$, we compute

$$\begin{aligned}
\frac{1}{2}\partial_t\{\|v_k^{(\alpha\beta)}\|^2\} &= \mathrm{Re}\,\langle \partial_t v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\rangle \\
&= \mathrm{Re}\,\langle f_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\rangle + \mathrm{Re}\,\langle a(x)D_x^{p-1}v_k^{(\alpha\beta)}, v_k^{\alpha\beta}\rangle \\
&\geq \underbrace{-\|f_k^{(\alpha\beta)}\|\|v_k^{(\alpha\beta)}\|}_{\text{negligible}} + \underbrace{\mathrm{Re}\,\langle a(x)D_x^{p-1}v_k^{(\alpha\beta)}, v_k^{\alpha\beta}\rangle}_{\text{important}}.
\end{aligned}$$

We decompose (choosing the constant $c_0 = 7^{-\sigma}/4^{p-1}$)

$$a(x)\xi^{p-1} = c_0\langle \nu_k^{p-1}\rangle^{-\sigma}\nu_k^{p-1} + \underbrace{I_{2,k}(x,\xi)}_{\substack{\text{negligible by}\\ \text{slow decay condition}}} + \underbrace{I_{3,k}(x,\xi)}_{\substack{\text{negligible by}\\ \text{support properties}}}$$

where

$$I_{2,k}(x,\xi) = \left(a(x)\xi^{p-1} - c_0\langle \nu_k^{p-1}\rangle^{-\sigma}\nu_k^{p-1}\right)\psi_k(x)\chi_k(\xi)$$

$$I_{3,k}(x,\xi) = \left(a(x)\xi^{p-1} - c_0\langle \nu_k^{p-1}\rangle^{-\sigma}\nu_k^{p-1}\right)(1-\psi_k(x)\chi_k(\xi)),$$

$$\chi_k(\xi) = h\left(\frac{\xi-\nu_k}{\frac{3}{4}\nu_k}\right), \quad \psi_k(x) = h\left(\frac{x-4\nu_k^{p-1}}{3\nu_k^{p-1}}\right)$$

The leading term in $\frac{1}{2}\partial_t\{\|v_k^{(\alpha\beta)}\|^2\} = \|v_k^{(\alpha\beta)}\|\partial_t v_k^{(\alpha\beta)}\|$ is so

$$\langle c_0\langle \nu_k^{p-1}\rangle^{-\sigma}\nu_k^{p-1}v_k^{(\alpha\beta)}, v_k^{\alpha\beta}\rangle \geq c_0 2^{-\sigma/2}\sigma_k^{(p-1)(1-\nu)}\|v_k^{(\alpha\beta)}\|^2.$$

Consequently, we have

$$\begin{aligned}\partial_t E_k(t) &= \sum_{\alpha \le N_k, \beta \le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \partial_t \|v_k^{(\alpha\beta)}(t,\cdot)\| \\ &\ge c_1 \nu_k^{(p-1)(1-\sigma)} E_k(t) - \text{lower order terms}\end{aligned}$$

Precisely, thanks to the choice of N_k, we get for k large enough

$$\partial_t E_k(t) \ge \frac{c_1}{2} \nu_k^{(p-1)(1-\sigma)} E_k(t) - C^{N+1} \nu_k^{C-cN}$$

$C, c > 0$ independent from k. By Gronwall inequality:

$$\begin{aligned}E_k(t) &\ge e^{\frac{c_1}{2}\nu_k^{(p-1)(1-\sigma)} t} \left\{ E_k(0) - C^{N+1} \nu_k^{C-cN} t \right\} \\ &\ge C e^{\frac{c_1}{2}\nu_k^{(p-1)(1-\sigma)} t} \left[\nu_k^{-(p+2)/2} e^{-c_{\rho_0} \nu_k^{\frac{1}{\theta}}} - t e^{-c \nu_k^{\frac{\lambda}{\theta_1}}} \right],\end{aligned}$$

provided that $\lambda < \min\{(p-1)(1-\sigma), 1\}$ and k being sufficiently large.

By a sharp choice of λ we get

$$E_k(t) \geq C_2 e^{\tilde{c}_0 \nu_k^{(p-1)(1-\sigma)}} e^{-\tilde{c}_{\rho_0} \nu_k^{\frac{1}{\theta}}} \geq C_3 e^{\frac{\tilde{c}_0}{2} \nu_k^{(p-1)(1-\sigma)}} \to \infty$$

and we have the contradiction.

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and we have the contradiction.
What can we say for a general operator of the form

$$D_t + a_p(t) D_x^p + \sum_{j=1}^{p} a_{p-j}(t,x) D_x^{p-j}$$

with $a_p \in C([0,T],\mathbb{R})$ and $a_{p-j} \in C([0,T],\mathcal{B}^\infty(\mathbb{R}))$?

Theorem 2 (Arias Jr, Ascanelli, C. 2023) Let $\theta > 1$. Let P an operator of the form

$$D_t + a_p(t) D_x^p + \sum_{j=1}^{p} a_{p-j}(t,x) D_x^{p-j}$$

with $a_p \in C([0,T];\mathbb{R})$ and $a_p(t) \neq 0 \,\forall t \in [0,T]$, and assume that the coefficients a_{p-j} satisfy the following conditions:

(i) there exist $R, A > 0$ and $\sigma_{p-j} \in [0,1]$, $j = 1, \ldots, p-1$, such that

$$Im\, a_{p-j}(t,x) \geq A\langle x\rangle^{-\sigma_{p-j}}, \quad x > R\,(\text{or } \ x < -R),\ t \in [0,T],\ j = 1, \ldots, p-1;$$

(ii) there exists $C > 0$ such that

$$|\partial_x^\beta a_{p-j}(t,x)| \leq C^{\beta+1}\beta!\langle x\rangle^{-\beta}, \quad x \in \mathbb{R},\ t \in [0,T], \quad j = 1, \ldots, p.$$

If the related Cauchy problem is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then

$$\Xi := \max_{j=1,\ldots,p-1}\{(p-1)(1-\sigma_{p-j}) - j + 1\} \leq \frac{1}{\theta}. \tag{7}$$

Remark. Let us notice that since $(p-1)(1-\sigma_{p-1}) > 0$ we always have $\Xi > 0$. We also point out the following inequalities

$$(p-1)(1-\sigma_{p-j}) - j + 1 \geq 1 \iff \sigma_{p-j} \leq \frac{p-1-j}{p-1},$$

$$(p-1)(1-\sigma_{p-j}) - j + 1 \leq 0 \iff \sigma_{p-j} \geq \frac{p-j}{p-1}.$$

Therefore, as a consequence of (7), when $\theta > 1$ we conclude the following:

- If $\sigma_{p-j} \leq \frac{p-1-j}{p-1}$ for some $j = 1, \ldots, p-1$, the Cauchy problem is not well-posed in $\mathcal{H}^\infty_\theta(\mathbb{R})$;
- If $\sigma_{p-j} \geq \frac{p-j}{p-1}$ for some $j = 1, \ldots, p-1$, then the power σ_{p-j} has no effect on the $\mathcal{H}^\infty_\theta$ well-posedness;
- If $\sigma_{p-j} \in \left(\frac{p-1-j}{p-1}, \frac{p-j}{p-1}\right)$ for some $j = 1, \ldots, p-1$, then the power σ_{p-j} imposes the restriction

$$(p-1)(1-\sigma_{p-j}) - j + 1 \leq \frac{1}{\theta}$$

for the indices θ where $\mathcal{H}^\infty_\theta$ well-posedness can be found.

Sufficient conditions for general p-evolution operators

Our next purpose is to find sufficient conditions for Gevrey well posedness for the Cauchy problem related to the operator

$$D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}(t,x)D_x^{p-j}$$

taking into account the restrictions on the decay rates we obtained.

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taking into account the restrictions on the decay rates we obtained. To achieve

the result we need to perform a change of variables of the form

$$Q_{\Lambda,k,\rho'}(t,x,\xi) = e^{\rho'\langle D\rangle_h^{1/\theta} + k(T-t)\langle D\rangle_h^{2(1-\sigma)}} \circ e^{\Lambda}(x,D)$$

with $\Lambda = \lambda_{p-1} + \lambda_{p-2} + \ldots + \lambda_2 + \lambda_1 \in S_\mu^{(p-1)(1-\sigma)}$ for some $\mu > 1$, $k > 0$, $\rho' \in (0,\rho)$ and $h \geq 1$ large enough. Each term λ_j should act on the terms of order j of the operator transforming them into the sum of a positive operator plus remainder terms whose order and regularity must be controlled.

Quasilinear p-evolution equations

After that we may extend our results to quasilinear p-evolution equations. We already prove such a result for $p = 3$, see

A. Arias Junior, A. Ascanelli, M.C., *KdV-type equations in projective Gevrey spaces*, J. Math. Pures Appl. 2023.

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THANK YOU FOR YOUR ATTENTION!