CR Geometry and Analysis

Howard Jacobowitz

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P &= \{z,w) \ : \ |z < 1, \quad |w| < 1\}
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There does not exist a biholomorphism

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The identity component of the group of automorphism of P leaving the origin fixed is given by

$$(z,w) \to e^{i\theta_1} z, e^{i\theta_2} w.$$

and is commutative.

Two elements in the identity component of the group of automorphism of B leaving the origin fixed are

$$\phi_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} 0 & e^{i\sigma} \\ -e^{-i\sigma} & 0 \end{pmatrix}.$$

For $0 < \sigma < \pi$, ϕ_1 and ϕ_2 do not commute.

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What are these invariants?

Élie Cartan

I resolved this question with an application of my general method of equivalence. The complete solution of Poincaré's problem led me to new geometric ideas.

For the real hypersurface

$$M = \{(z, w) : \phi(z, w) = 0\}$$

the complex vector field

$$L = \phi_{\bar{w}} \frac{\partial}{\partial \bar{z}} - \phi_{\bar{z}} \frac{\partial}{\partial \bar{w}}$$

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$$Q = \{(z,w) : \Im w = |z|^2\}$$

with $z = x + iy$ and $w = u + iv(x,y,u)$

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We will see this operator again.

In general, given a choice of L, choose a real-valued form ω and a complex-valued form ω_1

$$\omega(L) = 0, \quad \omega_1(L) = 0, \quad \omega \wedge \omega_1 \wedge \bar{\omega_1} \neq 0.$$

Normalize by $d\omega = i\omega_1 \wedge \bar{\omega_1} \ \ mod\ \omega$.

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$$\Omega = |\lambda|^2\omega \text{ and } \Omega_1 = \lambda(\omega_1 + \mu\omega)$$

These are well-defined forms on a bundle of fiber dimension 4 over M^3.

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These are well-defined forms on a bundle of fiber dimension 4 over M^3. Set

$$\Omega_2 = \frac{d\lambda}{\lambda} + A\omega_1 + B\bar{\omega_1} + C\omega$$

and

$$\Omega_3 = \frac{1}{\bar{\lambda}}(d\mu + D\omega_1 + E\bar{\omega_1} + F\omega).$$

Choose the coefficients to obtain

$$d\Omega = i\Omega_1 \wedge \bar{\Omega}_1 + (\Omega_2 + \bar{\Omega}_2) \wedge \Omega$$

and

$$d\Omega_1 = \Omega_2 \wedge \Omega_1 + \Omega_3 \wedge \Omega.$$

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Only

$$\rho = \Re C$$

remains undetermined. Set

$$\Omega_4 = \frac{1}{|\lambda|^2}\{d\rho + \dots\}$$

to obtain

$$
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d\Omega_2 &= 2i\Omega_1 \wedge \bar{\Omega_3} + i\bar{\Omega_1} \wedge \Omega_3 - \Omega \wedge \Omega_4 \\
d\Omega_3 &= -\Omega_1 \wedge \Omega_4 - \bar{\Omega_2} \wedge \Omega_3 - R\Omega \wedge \bar{\Omega_1} \\
d\Omega_4 &= i\Omega_3 \wedge \bar{\Omega_3} - (\Omega_2 + \bar{\Omega_2}) \wedge \Omega_4 - S\Omega \wedge \Omega_1 - \bar{S}\Omega \wedge \bar{\Omega_1}
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R and S are relative invariants.

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R and S are relative invariants. $R = 0$ on some open set implies that $S = 0$ on that open set and that there is a biholomorphism taking a possibly smaller open set of M^3 to the hyperquadric Q.

R is called the curvature of the CR structure. It is an invariant of order 6, not 9.

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Now for some analysis.

Uniformly the experience of the investigated type has shown that - speaking of existence in the local sense - there always were solutions, indeed smooth solutions, provided the equations were smooth enough. It was therefore a matter of considerable surprise to this author, to discover that this inference is in general erroneous. More precisely, there exist linear partial differential equations with coefficients in $\mathcal{C}^\infty$ which possess not a single smooth solution in any neighborhood.

Hans Lewy, *An example of a smooth linear partial differential equation without solution*, Annals of Mathematics 66 (1957).

Let $\phi(y_1)$ be a real-valued C^1 function.

Theorem

If there exists a C^1 solution to

$$\big(-(\partial/\partial x_1) - i(\partial/\partial x_2) + 2i(x_1 + ix_2)(\partial/\partial y_1)\big)u = 2\phi(y_1)$$

in a neighborhood of a point $(0, 0, y^*)$, then ϕ is analytic in some neighborhood of that point.

Set

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

and write

$$Lu = -u_{\bar{z}} + izu_{y_1}$$

Theorem

If $Lu = \phi(y_1)$ has a C^1 solution then ϕ is real analytic.

Let $w = y_1 + iy_2$. For $y_2 = |z|^2$, set

$$U(w, \bar{w}) = \int_{y_2 = constant} udz.$$

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This is natural if we consider $u(x_1, x_2, y_1)$ as a function on $Q = \{w = y_1 + i|z|^2\}$. Note that $U(y_1, 0) = 0$.
Write $z = re^{i\theta}$. So, $y_2 = r^2$ and

$$U(w, \bar{w}) = \int_0^{2\pi} ire^{i\theta}u(r, \theta, y_1)d\theta.$$

Claim

$$\frac{\partial U}{\partial \bar{w}} = \frac{1}{2}\int L u d\theta.$$

Assume the Claim.

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$V = U - 2\pi\psi$ is a holomorphic function of w in some set

$$\{a < y_1 < b, 0 < y_2 < c\}$$

and real on the real axis. The Reflection Principle applies and V is holomorphic near the y_1 axis. Thus ψ and ϕ are also real analytic as functions of y_1. So $Lu = f$ is not always locally solvable.

The proof of the Claim is an integration by parts.

$$\int_0^{2\pi} u e^{i\theta} d\theta = i \int_0^{2\pi} u_\theta e^{i\theta} d\theta.$$

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\begin{aligned}
2U_{\bar{w}} &= U_{y_1} + iU_{y_2} \\
&= \int u_{y_1} dz + i(i \int u_{\bar{z}} d\theta) \\
&= \int u_{y_1} izd\theta - \int u_{\bar{z}} d\theta \\
&= \int Lud\theta.
\end{aligned}
$$

A similar result holds for any strictly pseudo-convex hypersurface in $\mathbb{C}^2$: The associated linear partial derivative operator is not always solvable.

Abstract CR manifolds

(H,J) is an abstract CR structure on M^3

- $H \subset TM$ is a two-plane distribution
- J is an anti-involution on H

$$J : H \to H, \quad J^2 = -I$$

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For $M^3 \subset \mathbb{C}^2$ and $J_0 : \mathbb{C}^2 \to \mathbb{C}^2$

$$H_p = T_p(M) \cap J_0 T_p(M) = \text{the complex line tangent to } M \text{ at } p$$

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A complex vector field L on M^3 is an <u>abstract CR structure</u> if $\Re L$ and $\Im L$ are everywhere independent. Then

$$H = \{\Re L, \Im L\}.$$

and

$$JL = -iL.$$

Cartan's construction also applies to abstract CR manifolds.

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Does every homogeneous linear partial differential equation have a non-trivial solution?

Theorem (Nirenberg 1972)

There exists a smooth CR operator such that $Lh = 0$ in a neighborhood of some given point p implies that $dh(p) = 0$. In fact, h is a constant near p.

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2. There is an example of non-embeddability at a point of zero curvature (Pogorelov 1971).
3. M is locally embeddable near p if $K(p) = 0, \;\; \nabla K \neq O$ (Lin 1986).

Theorem (Jacobowitz and Treves, 1982, 1983)

Let L be the CR operator of any strictly pseudo-convex $M^3 \subset \mathbb{C}^2$ and let $p \in M^3$. There exists a complex vector field $\tilde{L}$ agreeing with L to infinite order at p such that $\tilde{L}h = 0$ in a neighborhood of p implies that $dh(p) = 0$. In fact, h is a constant near p.

Proof

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A local biholomorphic change of coordinates yields $M = \{v = \rho(z, \bar{z}, u)\}$ with $\rho(0) = 0, d\rho(0) = 0$ and

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Set $\tilde{L} = L + f\partial_z + g\partial_u$. We want to choose f, vanishing to infinite order at 0, such that $\tilde{L}h = 0$ implies $h_z(0) = 0$ and g such that $h_u(0) = 0$.

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Assume that for any neighborhood N of $0 \in \mathbb{R}^3$ there exist open sets $U \subset \Omega \subset N$, such that if $Lh = \phi$ in Ω and $supp(\phi) \subset U$, then

$$\iiint_U \phi dxdydu = 0.$$

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Next choose $N_j \to \{0\}$. Therefore Ω_j and $U_j \to \{0\}$ and so $h_z(0) = 0$.

Start with

$$M = \{v = |z|^2 + O(3; z, \bar{z}, u)\}.$$

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Let $S =$ a torus foliated by curves Γ_λ and T the solid torus.

Lemma

If $Lh = 0$ *in* $\Omega - T$, *then*

$$\int_{\Gamma_\lambda} h dz = 0$$

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We want to show that

$$\iiint_T Lhdxdydu = 0$$

if $supp(Lh) \subset T$.

Since $Lz = 0$ and $Lw = 0$, it follows that

$$d(hdzdw) = -Ludzd\bar{z}du.$$

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Thus

$$\begin{aligned} -\iiint_T (Lh)dzd\bar{z}du &= \iiint_T d(hdzdw) \\ &= \iint_S hdzdw \\ &= \int\int_{\Gamma_\lambda} hdzd\lambda \\ &= 0. \end{aligned}$$

Thus, if $Supp(f) \subset T_j$ and $f > 0$ in each T_j, then from

$$(L + f\partial_z)h = 0$$

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in a neighborhood of the origin we have that $h_z(0) = 0$. Using more open sets, an appropriate g, and a Baire category argument, we have

$$(L + f\partial_z + g\partial_u)h = 0$$

in a neighborhood of the origin implies h is a constant.

A CR embedding $f_0 : (M^3, V_0) \to \mathbb{C}^N$ is stable if

$$||V_1 - V_0|| \text{ small} \Rightarrow \exists f_1 : (M^3, V_1) \to \mathbb{C}^N \text{ with } ||f_1 - f_0|| \text{ small.}$$

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No CR embedding is stable.

Theorem (Lempert 1994)

Let M^3 be compact and (M^3, V_0) be strictly pseudo-convex. Let $f_0 : (M^3, V_0) \to \mathbb{C}^2$ be a CR embedding. If (M^3, V_1) has a CR embedding into some $\mathbb{C}^N$ then it has an embedding into $\mathbb{C}^2$ close to f_0.

Theorem (Caitlin, Lempert, 1992)

There exists a strictly pseudo-convex compact CR manifold in $\mathbb{C}^3$ *that is not stable.*

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Why this difference between $\mathbb{C}^2$ and $\mathbb{C}^3$? Reasonable from a geometric point of view.
(M^3, V_1) has a CR embedding into some $\mathbb{C}^N$ is equivalent to $\bar{\partial}_b$ has closed range on functions. What other condition is necessary to distinguish between $\mathbb{C}^2$ and $\mathbb{C}^3$?

Theorem (most likely) (Siqi Fu, Weixia Zhu)

- Let (M^3, V_t) be a smooth family of compact pseudo-convex CR manifolds of finite type. The existence of a uniform closed range estimate for $\bar{\partial}_b^t$ implies stability of the family.
- Let (M^3, V_t) be a smooth family of compact pseudo-convex CR manifolds of finite type. If the Kohn Laplacian $\Box_b^t$ acting on functions has a uniform spectral gap, then the family is stable.

Conjecture

Lempert's result holds for strictly pseudo-convex replaced by pseudoconvex of finite type.

THANK YOU