Mean-Dispersion Principles and the Wigner Transform

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Uncertainty principle

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Physical point of view: In quantum mechanics

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- the wave function of the position of a subatomic particle is the Fourier transform of the wave function of the momentum (mass times velocity)

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- the wave function of the position of a subatomic particle is the Fourier transform of the wave function of the momentum (mass times velocity)
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- any attempt to measure precisely the velocity of the particle will knock it about in an unpredictable way, so that a simultaneous measurement of its position has no validity
- if one forces a particle to be in a small region, then we cannot measure its velocity

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From the mathematical point of view

Fourier transform: For $f \in L^2(\mathbb{R})$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

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Depending on the definition of "concentration", one gets different uncertainty principles.

Heisenberg's uncertainty principle

For $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2} = 1$ we measure "concentration" in terms of the standard deviation

$$\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 dx$$

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Consider the analogous quantity for $\hat{f}$:

$$\int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 d\xi$$

Note that also $\|\hat{f}\|_{L^2} = 1$ by Plancharel.

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Heisenberg's uncertainty principle

No matter which point $b \in \mathbb{R}$ we choose, the support of $\hat{f}$ cannot be concentrated around b if the support of f is concentrated around a.

A more precise mathematical formulation

Heisenberg's uncertainty principle

Let $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2} = 1$. Then

$$\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 dx \cdot \int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 d\xi \geq 1$$

and equality holds iff f is a Gaussian, up to translations and modulations: $f(x) = ce^{ibx} \cdot e^{-\gamma(x-a)^2}$ for some $c \in \mathbb{C}, \gamma > 0$

A second mathematical formulation

Let $f \in L^2(\mathbb{R}) \setminus \{0\}$,

mean associated to f: $\mu(f) := \frac{1}{\|f\|_{L^2}^2} \int_{\mathbb{R}} t|f(t)|^2 dt$

dispersion associated to f: $\Delta(f) := \frac{1}{\|f\|_{L^2}} \left(\int_{\mathbb{R}} (t - \mu(f))^2 |f(t)|^2 dt \right)^{1/2}$

(mimimum of the standard deviation)

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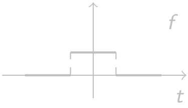
From the quantum mechanics point of view

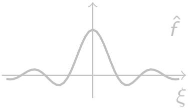
Position and velocity of a subatomic particle cannot be simultaneously well predicted

Benedicks's uncertainty principle

For $f \in L^1(\mathbb{R}) \setminus \{0\}$ think as "concentration" in terms of "living" entirely on a set of finite measure.



time limited



not band limited

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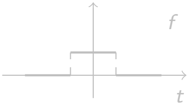

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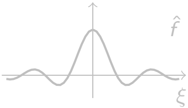

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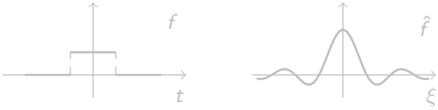
Note: If f is a C^∞ function (or distribution) with compact support, this was already known by the Paley-Wiener Theorem, since its Fourier transform is an analytic function, hence the zeros of $\hat{f}$ are isolated.



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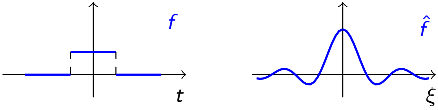
Physical interpretation

A non-zero signal cannot both be time limited and band limited



time limited | not band limited

Donoho-Stark uncertainty principle

ε-concentration on a measurable set

$f \in L^2(\mathbb{R}) \setminus \{0\}$ is ε_T-concentrated on a measurable set $T \subseteq \mathbb{R}$ if

$$\left(\int_{\mathbb{R}\setminus T} |f(x)|^2 dx \right)^{1/2} \leq \varepsilon_T \|f\|_{L^2(\mathbb{R})}$$

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Let $f \in L^2(\mathbb{R}) \setminus \{0\}$. If f is ε_T-concentrated on T and $\hat{f}$ is ε_Ω-concentrated on Ω, then

$$m(T) \cdot m(\Omega) \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2.$$

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Corollary

If $f \in L^2(\mathbb{R}) \setminus \{0\}$ with $\operatorname{supp} f \subseteq T$ and $\operatorname{supp} \hat{f} \subseteq \Omega$ then
$m(T) \cdot m(\Omega) \geq 1$

since $\varepsilon_T = \varepsilon_\Omega = 0$

Hardy's uncertainty principle

Hardy's Uncertainty Principle

Let $a, b, C, N > 0$ and $f \in L^2(\mathbb{R})$ s.t. for almost all $x, \xi \in \mathbb{R}$,

$$|f(x)| \leq C(1+|x|)^N e^{-a\pi x^2} \quad \text{and} \quad |\hat{f}(\xi)| \leq C(1+|\xi|)^N e^{-b\pi \xi^2}. \tag{1}$$

- If $a \cdot b > 1$ then $f = 0$.
- If $a \cdot b = 1$ then $f(x) = P(x)e^{-a\pi x^2}$ for a polynomial P of degree $\leq N$.

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Beurling-Hörmander (1991), case $a \cdot b > 1$

For $f \in L^1(\mathbb{R})$, we have

$$\iint |f(x)\widehat{f}(\xi)|e^{2\pi|x\xi|}dxd\xi < \infty \implies f = 0.$$

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Demange (2006), case $a \cdot b < 1$

The class of L^2-functions satisfying (1) is infinite-dimensional. However, it cannot contain an infinite orthonormal sequence.

Two Uncertainty Principles due to Shapiro

Shapiro's Umbrella Theorem

Let $\varphi, \psi \in L^2(\mathbb{R})$. If $\{e_k\} \subset L^2(\mathbb{R})$ is an orthonormal sequence of functions s.t. for all k and almost all $x, \xi \in \mathbb{R}$,

$$|e_k(x)| \leq |\varphi(x)| \quad \text{and} \quad |\widehat{e_k}(\xi)| \leq |\psi(\xi)|,$$

then the sequence $\{e_k\}$ is finite.

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Shapiro's mean-dispersion principle (Shapiro, unpublished manuscript, 1991)

There does not exist an infinite orthonormal sequence $\{f_k\}_{k\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$ such that means and dispersions $\mu(f_k)$, $\mu(\hat{f}_k)$, $\Delta(f_k)$, $\Delta(\hat{f}_k)$ are all uniformly bounded

Quantitative versions of Shapiro's Uncertainty Principles

With the use of Rayleigh-Ritz techniques:

Sharp mean-dispersion principle (Jaming-Powell, JFA, 2007)

Let $\{f_k\}_{k\in\mathbb{N}_0}$ be an orthonormal sequence in $L^2(\mathbb{R})$. Then for all $n \in \mathbb{N}_0$

$$\sum_{k=0}^{n}(\Delta^2(f_k)+\Delta^2(\hat{f}_k)+\mu^2(f_k)+\mu^2(\hat{f}_k)) \geq (n+1)^2$$

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Using geometric combinatorics:

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then N is bounded by a quantity that depends on certain geometric properties of φ and ψ.

Uncertainty principles and time-frequency analysis

In [BJO][2] we consider uncertainty principles of mean-dispersion type involving time-frequency representations (in particular, the Wigner distribution).

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Motivation

Time-frequency analysis combines the features of f and $\hat{f}$ into a single function, a so-called time-frequency representation

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There are several type of time-frequency representations. But uncertainty principles cannot be avoided and each time-frequency representation entails its own peculiar version of the uncertainty principle.

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Cross-Wigner distribution

For $f, g \in L^2(\mathbb{R})$,

$$W(f,g)(x,\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-it\xi} dt, \qquad x, \xi \in \mathbb{R}.$$

Set $W(f) = W(f,f)$.

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Hermite functions

A particular orthonormal basis of $L^2(\mathbb{R})$:

Hermite functions

$$h_k(t) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-t^2/2} H_k(t), \qquad t \in \mathbb{R},\ k \in \mathbb{N}_0,$$

where H_k is the *Hermite polynomial* of degree k given by

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- $\{h_{j,k} := \mathcal{F}^{-1} W(h_j, h_k)\}_{j,k \in \mathbb{N}_0}$ form an orthonormal basis of $L^2(\mathbb{R}^2)$

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- $\{h_k\}_{k \in \mathbb{N}_0}$ form an orthonormal basis of $L^2(\mathbb{R})$
- h_k are in the Schwartz class $\mathcal{S}(\mathbb{R})$ (and form an absolute basis of $\mathcal{S}$)
- $\{h_{j,k} := \mathcal{F}^{-1} W(h_j, h_k)\}_{j,k \in \mathbb{N}_0}$ form an orthonormal basis of $L^2(\mathbb{R}^2)$
- $h_{j,k}$ are eigenfunctions of the *twisted Laplacian*:

$$L h_{j,k}(y,t) = (2k+1) h_{j,k}(y,t), \qquad \forall j,k \in \mathbb{N}_0$$

where $L := \left(D_y - \frac{1}{2} t\right)^2 + \left(D_t + \frac{1}{2} y\right)^2$

Further properties of Hermite functions

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Uncertainty principle for the Wigner transform

Theorem (Boiti-Jornet-Oliaro)

Let $\{f_k\}_{k\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$ with $\|f_k\| = 1$ and $\{g_k\}_{k\in\mathbb{N}_0}$ an orthonormal sequence in $L^2(\mathbb{R})$. Then

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Moreover, $\langle \hat{L}W(f_i,g_k),W(f_i,g_k)\rangle$ uniformly bounded $\Rightarrow$ $\{g_k\}$ must be finite (while $\{f_k\}$ may be infinite).

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Proof: $\{\hat{h}_{j,k}\}=\{W(h_j,h_k)\}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$:

$$W(f_i,g_k)=\sum_{j,\ell=0}^{+\infty} c_{j,\ell}^{(i,k)}W(h_j,h_\ell)$$

with

$$c_{j,\ell}^{(i,k)}=\langle W(f_i,g_k),W(h_j,h_\ell)\rangle=\langle f_i,h_j\rangle\overline{\langle g_k,h_\ell\rangle}$$

Proof

Then, from $\hat{L}\hat{h}_{j,\ell} = (2\ell+1)\hat{h}_{j,\ell}$:

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with $0 \le \alpha_\ell \le \|h_\ell\|^2 = 1$ and $\sum_{\ell=0}^{+\infty}\alpha_\ell = \sum_{k=0}^{n}\underbrace{\sum_{\ell=0}^{+\infty}|\langle g_k, h_\ell\rangle|^2}_{\|g_k\|^2=1} = n+1$

Proof

$$n + 1 = \underbrace{\alpha_0}_{\leq 1} + \ldots + \underbrace{\alpha_n}_{\leq 1} + \underbrace{\sum_{\ell \geq n+1} \alpha_\ell}_{=: R_n}$$

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Mean-dispersion principle for the Wigner transform

Corollary (Boiti-Jornet-Oliaro) (same proof with $f_k = g_k$)

If $\{f_k\}_{k\in\mathbb{N}_0}$ is an orthonormal sequence in $L^2(\mathbb{R})$, then

$$\sum_{k=0}^{n} \langle \hat{L}W(f_k), W(f_k)\rangle \geq (n+1)^2, \qquad \forall n \in \mathbb{N}_0, \tag{1}$$

and equality holds $\forall n \leq n_0$ iff $f_k = c_k h_k$, $c_k \in \mathbb{C}$, $|c_k| = 1$, $0 \leq k \leq n_0$.

[3] P. Jaming, A.M. Powell, *Uncertainty principles for orthonormal sequences*, J. Funct. Anal. **243** (2007), 611-630

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Remark

(1) may be interpreted as a mean-dispersion principle since

$$\langle \hat{L}W(f_k), W(f_k)\rangle = \Delta^2(f_k) + \Delta^2(\hat{f}_k) + \mu^2(f_k) + \mu^2(\hat{f}_k)$$

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We have thus provided an elementary proof of Shapiro's mean-dispersion principle (Jaming-Powell use in [JP][3] the Rayleight-Ritz technique to estimate eigenvalues of operators).

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Different operators

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- $\langle D_j W(f), W(f)\rangle = 0, \quad j = 1, 2$
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Example

$\forall n \in \mathbb{N}_0$:

$$\sum_{k=0}^{n} \int (x^2 + \xi^2)|W(f_k)|^2 dx d\xi \geq \frac{(n+1)^2}{2}$$

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N.B. If $\mu(f_k) = \mu(\hat{f}_k) = 0$ then $\int (x^2+\xi^2)|W(f_k)|^2 dx d\xi$ is the trace of the covariance matrix associated to $|W(f_k)|$.

Mean-dispersion principle for the Cohen class

Cohen class

$$Q(f,g) = \frac{1}{\sqrt{2\pi}}\sigma * W(f,g), \qquad f,g \in \mathcal{S}, \sigma \in \mathcal{S}'$$

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Theorem (Boiti-Jornet-Oliaro)

Let $\{f_k\}_{k\in\mathbb{N}_0}, \{g_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ be two orthonormal sequences in $L^2(\mathbb{R})$. Then

$$\sum_{k=0}^{n} \langle \tilde{L} Q(f_j, g_k), Q(f_j, g_k)\rangle \geq (n+1)^2, \qquad \forall j, n \in \mathbb{N}_0,$$

for any linear partial differential operator $\tilde{L}$ of the form

$$\tilde{L}(M_1, M_2, D_1, D_2) = \left(M_1 + \frac{1}{2}D_2 - P_1\right)^2 + \left(\frac{1}{2}D_1 - M_2 + P_2\right)^2$$

with $\quad P_1 = (iD_1 P)(D_1, D_2), \quad P_2 = (iD_2 P)(D_1, D_2), \quad P \in \mathbb{R}[\xi, \eta],$

$$\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}), \quad Q(f_j, f_k) = \frac{1}{\sqrt{2\pi}}\sigma * W(f_j, f_k)$$

Mean-dispersion principle for the Cohen class

Cohen class

$$Q(f,g) = \frac{1}{\sqrt{2\pi}}\sigma * W(f,g), \qquad f,g \in \mathcal{S}, \sigma \in \mathcal{S}'$$

Theorem (Boiti-Jornet-Oliaro)

Let $\{f_k\}_{k\in\mathbb{N}_0}, \{g_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ be two orthonormal sequences in $L^2(\mathbb{R})$. Then

$$\sum_{k=0}^{n} \langle \tilde{L}Q(f_j,g_k), Q(f_j,g_k)\rangle \geq (n+1)^2, \qquad \forall j,n \in \mathbb{N}_0,$$

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Idea of the proof:

$\langle \tilde{L}Q(f,g), Q(f,g)\rangle = \mu^2(g) + \mu^2(\hat{g}) + \Delta^2(g) + \Delta^2(\hat{g})$

Two concrete examples

Example 1

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Now, we consider the operator $P(M_1, M_2) = M_1^2 + M_2^2$, and by direct computations

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Then, in this case, we have, $\forall n \in \mathbb{N}_0$,

$$\sum_{k=1}^{n} \langle ((M_1 - P_1)^2 + (M_2 - P_2)^2) Q(f_j, f_k), Q(f_j f_k) \rangle \geq \frac{(n+1)^2}{2}.$$

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If $\{u_k\}_{k\in\mathbb{N}_0}, \{v_k\}_{k\in\mathbb{N}_0}$ are Riesz bases for $L^2(\mathbb{R})$, then

$$\sum_{k=0}^{n}\langle \hat{L}W(u_i,v_k), W(u_i,v_k)\rangle \geq \frac{\|U_2^{-1}\|^2}{\|U_1\|^2}\left[\frac{n+1}{\|U_2^{-1}\|^2\|U_2\|^2}\right]^2$$

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Corollary (Boiti-Jornet-Oliaro)

If $\{u_k\}_{k\in\mathbb{N}_0}$ is a Riesz basis for $L^2(\mathbb{R})$ with $U(u_k)=h_k$, then

$$\sum_{k=0}^{n}(\Delta^2(u_k)+\Delta^2(\hat{u}_k)+\mu^2(u_k)+\mu^2(\hat{u}_k)) \geq \frac{1}{\|U^{-1}\|^2\|U\|^2}\left[\frac{n+1}{\|U^{-1}\|^2\|U\|^2}\right]^2$$

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Remark: If $\{u_k\}$ orthonormal then $\|U\|=\|U^{-1}\|=1$ and we obtain again Shapiro's mean-dispersion principle with the same estimate $(n+1)^2$ ([JP]: $\frac{1}{2}(n+1)(2n+1)$)

Main references

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Thank you for your attention!