

Analytic hypoellipticity in the chiral model of twisted bilayer graphene

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Joint work with M. Zworski

Introduction. The chiral model of twisted bilayer graphene

In this talk, we shall be concerned with some aspects of semiclassical analysis for a class of **non-self-adjoint operators** coming from **condensed matter physics of 2D materials**.

The chiral model of TBG

G. Tarnopolsky – A. Kruchkov – A. Vishwanath (2019) :

$$H(\alpha) := \begin{pmatrix} 0 & D(\alpha)^* \\ D(\alpha) & 0 \end{pmatrix}, \quad D(\alpha) := \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix},$$

$$z = x_1 + ix_2 \in \mathbb{C}, \quad D_{\bar{z}} = \frac{1}{i}\partial_{\bar{z}} = \frac{1}{2i}(\partial_{x_1} + i\partial_{x_2}),$$

acting on $L^2(\mathbb{C}; \mathbb{C}^4)$. Here $U(z)$ is the **Bistritzer–MacDonald potential**,

$$U(z) = -iK \sum_{\ell=0}^2 \omega^\ell e^{i\langle z, \omega^\ell K \rangle}, \quad K = \frac{4\pi}{3}, \quad \omega = e^{2\pi i/3}, \quad \langle z, w \rangle = \operatorname{Re}(z\bar{w}).$$

The Hamiltonian $H(\alpha)$ is derived from the full [Bistritzer–MacDonald \(2011\)](#) Hamiltonian by removing certain tunneling interactions between the two sheets of graphene. The dimensionless coupling constant α is such that the [angle of twisting](#) $\asymp 1/\alpha$.

Mathematical derivation :

[Cancès-Garrigue-Gontier \(2023\)](#), [Watson-Kong-MacDonald-Luskin \(2023\)](#).

Let

$$\Lambda := \omega\mathbb{Z} \oplus \mathbb{Z}, \quad \Lambda^* = \frac{4\pi i}{\sqrt{3}}\Lambda.$$

Here $\Lambda^* = \{k \in \mathbb{R}^2; \langle k, \gamma \rangle \in 2\pi\mathbb{Z} \text{ for every } \gamma \in \Lambda\}$ is the [dual lattice](#).

[Symmetries](#) of the potential U :

$$U(z + \gamma) = e^{i\langle \gamma, K \rangle} U(z), \quad \gamma \in \Lambda, \quad U(\omega z) = \omega U(z), \quad \overline{U(\bar{z})} = -U(-z),$$

$\implies U$ is [periodic](#) with respect to $\Gamma = 3\Lambda$.

Flat bands

Performing a **Floquet reduction** of $H(\alpha)$, we are led to consider the family

$$H_k(\alpha) := e^{i\langle z, k \rangle} H(\alpha) e^{-i\langle z, k \rangle} = \begin{pmatrix} 0 & D(\alpha)^* - \bar{k} \\ D(\alpha) - k & 0 \end{pmatrix}, \quad k \in \mathbb{C}/\Gamma^*,$$

acting on $L^2(\mathbb{C}/\Gamma; \mathbb{C}^4)$, with the domain $H^1(\mathbb{C}/\Gamma; \mathbb{C}^4)$. Here Γ^* is the dual lattice of Γ . A **flat band** at zero energy for $H(\alpha)$ occurs when

$$0 \in \text{Spec}_{L^2(\mathbb{C}/\Gamma)}(H_k(\alpha))$$

for all $k \in \mathbb{C}$, or equivalently, when

$$\text{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) = \mathbb{C}.$$

We have

$$D(\alpha) : H^1(\mathbb{C}/\Gamma; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Gamma; \mathbb{C}^2), \quad \alpha \in \mathbb{C},$$

is **Fredholm of index 0** such that

$$\text{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) = \text{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) + k, \quad k \in \Gamma^*.$$

The spectrum of $D(\alpha)$ and magic angles

Theorem (S. Becker, M. Embree, J. Wittsten, and M. Zworski (2022))

There exists a discrete set $\mathcal{A} \subset \mathbb{C}$ such that

$$\text{Spec}_{L^2(\mathbb{C}/\Gamma)} D(\alpha) = \begin{cases} \Gamma^*, & \alpha \notin \mathcal{A}, \\ \mathbb{C}, & \alpha \in \mathcal{A}. \end{cases}$$

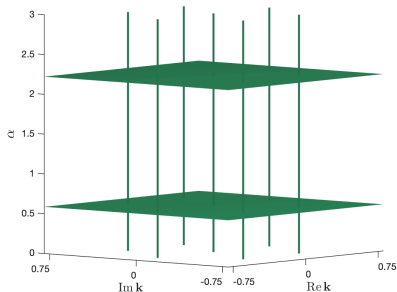


FIGURE – Spectrum of $D(\alpha)$ as α varies. Magic angles : $1/\alpha$, $\alpha \in \mathcal{A}$.

Crucial component of the proof : [symmetry protected eigenstates](#) at 0,

$$\text{Ker } L^2_{\rho_1,0}(\mathbb{C}/\Gamma)D(\alpha) \neq \{0\}, \quad \alpha \in \mathbb{C}.$$

J. Galkowski – M. Zworski (2023) : an abstract formulation of the flat band condition.

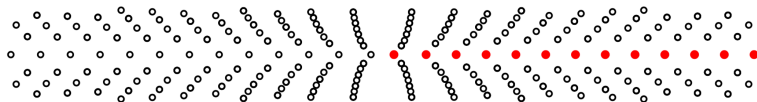


FIGURE – Reciprocals of magic angles for the Bistritzer-MacDonald potential (Becker–Embree–Wittsten–Zworski (2022)).

S. Becker – T. Humbert – M. Zworski (2023) : the set \mathcal{A} is [infinite](#).

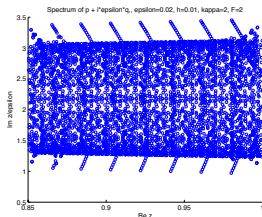
A. Watson – M. Luskin (2021), S. Becker – T. Humbert – M. Zworski (2023) : existence of the first [real](#) positive magic α .

Quantization condition for magic angles ?

Numerical observation by Tarnopolsky – Kruchkov – Vishwanath (2019),
Becker – Embree – Wittsten – Zworski (2022) : if $\alpha_1 < \alpha_2 < \dots < \alpha_j < \dots$
is the sequence of all real α 's in \mathcal{A} , then

$$\alpha_{j+1} - \alpha_j \simeq 1.515, \quad j \leq 13.$$

A. Melin – J. Sjöstrand (2002), J. Sjöstrand – M.H. (2004 – 2018) :
quantization rules for eigenvalues of semi-classical non-self-adjoint **analytic**
operators in dimension 2.



Can we apply the 2D non-self-adjoint machinery in this setting?

Spectra of elliptic first order scalar operators on tori

A. Melin – J. Sjöstrand (2002) : let

$$P = a(z)2D_{\bar{z}} + b(z)$$

on $L^2(\mathbb{C}/\Gamma)$, with $a, b \in C^\infty(\mathbb{C}/\Gamma)$, a nowhere vanishing. We have :

$$\lambda \in \text{Spec}_{L^2(\mathbb{C}/\Gamma)}(P) \iff \mathcal{F} \left(\frac{b}{a} \right) (0) - \lambda \mathcal{F} \left(\frac{1}{a} \right) (0) \in \Gamma^*.$$

In particular, we get a lattice of eigenvalues precisely when

$$\mathcal{F} \left(\frac{1}{a} \right) (0) \neq 0,$$

whereas if $\mathcal{F} (1/a) (0) = 0$, we get

$$\text{Spec}_{L^2(\mathbb{C}/\Gamma)} P = \begin{cases} \mathbb{C}, & \mathcal{F} (b/a) (0) \in \Gamma^*, \\ \emptyset, & \mathcal{F} (b/a) (0) \notin \Gamma^*. \end{cases}$$

R. Seeley (1986) : a similar example in 1D, $P(\alpha) = e^{ix} D_x + \alpha e^{ix}$,
 $x \in \mathbb{R}/2\pi\mathbb{Z}$.

Protected states in the semiclassical limit

This talk : Understand the structure of protected eigenstates at 0 of $D(\alpha)$ in the **small angle limit** $0 < \alpha \rightarrow \infty$ (within or without the magic set),

$$D(\alpha)u = 0, \quad u \in L^2_{\rho_1,0}(\mathbb{C}/\Gamma; \mathbb{C}^2).$$

Semiclassical formulation with $0 < h = \frac{1}{\alpha} \ll 1$,

$$p(x, hD_x)u = 0, \quad p(x, hD_x) = hD(\alpha) = \begin{pmatrix} 2hD_{\bar{z}} & U(z) \\ U(-z) & 2hD_{\bar{z}} \end{pmatrix},$$

$$p(x, \xi) = \begin{pmatrix} 2\bar{\zeta} & U(z) \\ U(-z) & 2\bar{\zeta} \end{pmatrix}, \quad z = x_1 + ix_2, \quad \zeta = \frac{1}{2}(\xi_1 - i\xi_2).$$

Principally scalar reduction

Observe that

$$(hD(-\alpha))(hD(\alpha)) = q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x),$$

where

$$q(x, \xi) = \det p(x, \xi) = 4\bar{\xi}^2 - U(z)U(-z),$$

$$q(x, hD_x) = (2hD_{\bar{z}})^2 - U(z)U(-z),$$

and

$$R(x) = \begin{pmatrix} 0 & 2D_{\bar{z}}U(z) \\ -D_{\bar{z}}U(-z) & 0 \end{pmatrix},$$

to get

$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x)) u = 0.$$

Classically forbidden regions

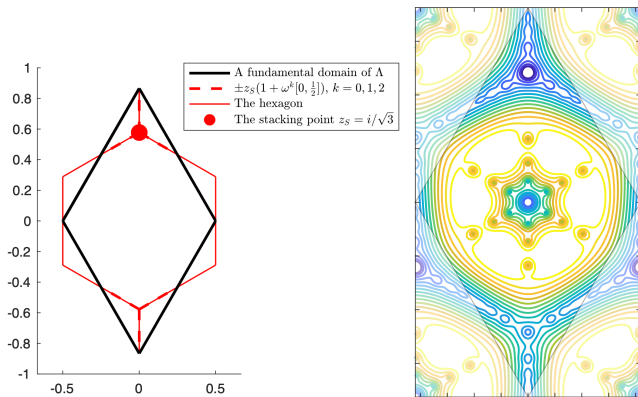


FIGURE – Left : the vertices of the hexagon in a fundamental domain of Λ are given by the *stacking points* $\pm z_S$, $z_S = i/\sqrt{3}$, i.e. points of high symmetry satisfying $\pm \omega z_S \equiv \pm z_S \pmod{\Lambda}$. Right : plot of $\log |u(z, \alpha)|$ where u is the protected state in the kernel of $D(\alpha)$ on $H^1(\mathbb{C}/\Gamma)$ and $\alpha = 11.345$. Dark blue corresponds to $|u| \simeq 10^{-7}$ and yellow to $|u| \simeq 1$: we see **exponential decay** $|u(z, \alpha)| \leq e^{-\alpha_0/h}$ near the **hexagon** and near its **center**.

The Poisson bracket $\{q, \bar{q}\}$

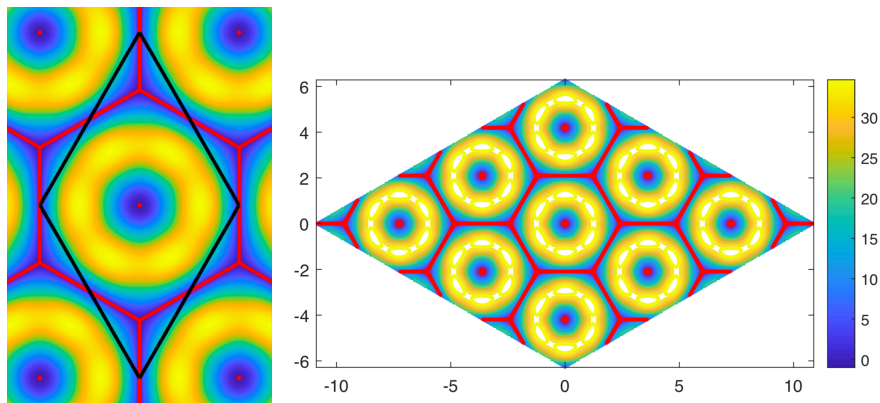


FIGURE – Left : the contour plot of $|\{q, \bar{q}\}|_{q^{-1}(0)}$, for q given by the determinant of the semiclassical symbol of $hD(\alpha)$, $\alpha = 1/h$, $q(x, \xi) = (2\bar{\zeta})^2 - U(z)U(-z)$. Right : the contour plot of $|\{q, \bar{q}\}|_{q^{-1}(0)}$ over a fundamental domain of $\Gamma = 3\Lambda$. The set where $\{q, \bar{q}\}|_{q^{-1}(0)} = 0$ is in red.

Classically forbidden regions for Schrödinger operators

Let

$$(-h^2\Delta + V(x) - E)u = 0, \quad x \in \mathbb{R}^n.$$

Exponential decay of eigenfunctions in the **classically forbidden region** $\mathcal{U} = \{x \in \mathbb{R}^n; V(x) > E\}$ is a consequence of **ellipticity** :

$$p(x_0, \xi) = \xi^2 + V(x_0) - E \neq 0, \quad x_0 \in \mathcal{U}, \quad \xi \in \mathbb{R}^n.$$

L. Lithner (1964), S. Agmon (1982), B. Simon (1984), B. Helffer – J. Sjöstrand (1984).

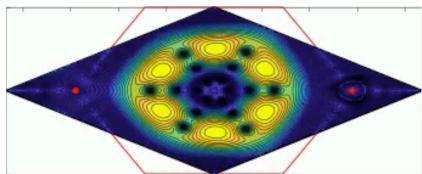
For $q(x, \xi) = (2\bar{\zeta})^2 - U(z)U(-z)$, there are **no** classically forbidden regions in the sense of ellipticity,

$$\forall x_0 \in \mathbb{R}^2 \quad q^{-1}(0) \cap \pi^{-1}(x_0) = \{\xi \in \mathbb{R}^2; q(x_0, \xi) = 0\} \neq \emptyset.$$

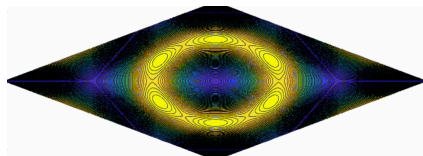
Here $\pi : T^*\mathbb{R}^2 \ni (x, \xi) \mapsto x \in \mathbb{R}^2$ is the natural projection.

Use **(analytic) hypoellipticity as a replacement?**

Classically forbidden regions when $\{q, \bar{q}\} = 0$



$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x))u = 0$$



$$|\{q, \bar{q}\}|_{q^{-1}(0)}$$

Theorem (M. Zworski – M.H. 2023)

Let $U \subset \mathbb{R}^2$ be open and let

$$P = P(x, hD_x; h) = Q \otimes 1_{\mathbb{C}^2} + h \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \quad x \in U,$$

be a *principally scalar* system of semiclassical differential operators with *real analytic* coefficients in U , such that $Q = q(x, hD_x)$ is *classically elliptic* of order 2, and $R_{k\ell} = R_{k\ell}(x, hD_x)$ are of order 1, for $1 \leq k, \ell \leq 2$. Assume that for $x_0 \in U$, we have

$$\{q, \bar{q}\}|_{q^{-1}(0) \cap \pi^{-1}(x_0)} = 0, \quad \{q, \{q, \bar{q}\}\}|_{q^{-1}(0) \cap \pi^{-1}(x_0)} \neq 0,$$

and $H_{\operatorname{Re} q}$ and $H_{\operatorname{Im} q}$ are linearly independent on $q^{-1}(0) \cap \pi^{-1}(x_0)$. If $Pu = 0$ in U and $\|u\|_{L^2(U)} \leq \mathcal{O}(1)$, then there exists an h -independent neighborhood Ω of x_0 and $C_0, c_0 > 0$ such that for all $0 < h \leq h_0$ we have,

$$|u(x; h)| \leq C_0 e^{-c_0/h}, \quad x \in \Omega.$$

Related recent work : J. Sjöstrand – M. Vogel (2023) (fine tunneling estimates for a model operator).

Example. Let

$$q(x, \xi) = \xi + ix^2, \quad (x, \xi) \in T^*\mathbb{R},$$

with $x_0 = 0$. Then

$$\{q, \bar{q}\}(0, 0) = 0, \quad \{q, \{q, \bar{q}\}\} = -4i \neq 0,$$

so the bracket conditions hold.

If

$$0 = q(x, hD_x)u = \frac{h}{i} \left(\partial_x - \frac{x^2}{h} \right) u,$$

then

$$u(x; h) = u(0, h)e^{x^3/3h}.$$

For this to be uniformly bounded near 0, we need $u(0; h) = \mathcal{O}(1)e^{-c/h}$, $c > 0$, and hence $|u(x; h)| \leq e^{-c/2h}$ for $|x|$ small.

Some words about the proof

Step I. Establish **microlocal** exponential decay of u .

Step II. From microlocal to local exponential decay.

When describing Step I, we need to recall the notion of the **semiclassical analytic wave front set** of an h -tempered family $h \mapsto u(h) \in \mathcal{D}'(U)$.

The key role is played by the FBI (**Fourier-Bros-Iagolnitzer**) – Bargmann transform,

$$T_h w(x) = \int e^{\frac{i}{h}\varphi_0(x,y)} w(y) dy, \quad \varphi_0(x,y) = \frac{i}{2}(x-y)^2.$$

Given $(y_0, \eta_0) \in T^*U$, we have $(y_0, \eta_0) \notin \text{WF}_{a,h}(u)$ precisely when $\exists \delta > 0, C > 0, V = \text{neigh}(y_0 - i\eta_0, \mathbb{C}^2)$ such that

$$|T_h(\chi u)(x)| \leq C e^{(\Phi_0(x) - \delta)/h}, \quad x \in V, \quad 0 < h \leq h_0.$$

Here $\chi \in C_0^\infty(U)$, $\chi(y) = 1$ in a neighborhood of y_0 , and

$$\Phi_0(x) := \frac{1}{2} |\text{Im } x|^2.$$

FBI transforms are flexible objects, and in the proof we work with a **local transform** of the form

$$T_h u(x) = \int e^{i\varphi(x,y)/h} a(x,y;h) \chi(y) u(y), \quad x \in \text{neigh}(x_0, \mathbb{C}^2).$$

Here $\chi \in C_0^\infty(U)$, $\chi = 1$ near y_0 , and $\varphi \in \text{Hol}(\text{neigh}((x_0, y_0), \mathbb{C}^4))$, for some $x_0 \in \mathbb{C}^2$, is such that

$$-\varphi'_y(x_0, y_0) = \eta_0, \quad \text{Im } \varphi''_{yy}(x_0, y_0) > 0, \quad \det \varphi''_{xy}(x_0, y_0) \neq 0.$$

The amplitude $a(x, y; h)$ is an **elliptic classical analytic symbol** in a neighborhood of (x_0, y_0) ,

$$a(x, y; h) \sim \sum_{j=0}^{\infty} a_j(x, y) h^j,$$

where a_j are holomorphic, with $a_0 \neq 0$, and such that

$$|a_j(x, y)| \leq C^{j+1} j^j, \quad j = 0, 1, 2, \dots$$

J. Sjöstrand (1982).

The FBI transform $T_h u$ of u is holomorphic and satisfies for each $\varepsilon > 0$,

$$|T_h u(x)| \leq \mathcal{O}_\varepsilon(1) e^{(\Phi(x)+\varepsilon)/h}, \quad x \in \text{neigh}(x_0, \mathbb{C}^2),$$

where the weight

$$\Phi(x) = \sup_{y \in \text{neigh}(y_0, \mathbb{R}^2)} (-\text{Im } \varphi(x, y))$$

is **strictly plurisubharmonic**.

The definition of $\text{WF}_{a,h}(u)$ is **independent** of the choice of an FBI transform :

Theorem (J. Sjöstrand, 1982)

We have $(y_0, \eta_0) \notin \text{WF}_{a,h}(u)$ if and only if there exist $\delta > 0$, $C > 0$, and $V = \text{neigh}(x_0, \mathbb{C}^2)$ such that

$$|T_h u(x)| \leq C e^{(\Phi(x)-\delta)/h}, \quad x \in V, \quad 0 < h \leq h_0.$$

Microlocal analytic hypoellipticity

Proposition

Let $(0, h_0] \ni h \mapsto u(h) \in \mathcal{D}'(U; \mathbb{C}^2)$ be h -tempered. If at some point $\rho = (y_0, \eta_0) \in q^{-1}(0)$ we have

$$\{q, \bar{q}\}(\rho) = 0, \quad \{q, \{q, \bar{q}\}\}(\rho) \neq 0, \quad H_q(\rho) \not\parallel H_{\bar{q}}(\rho), \quad \rho \notin \text{WF}_{a,h}(Pu),$$

then $\rho \notin \text{WF}_{a,h}(u)$.

This result is based on the work of M. Kashiwara and T. Kawai (1979), J.-M. Trépreau (1984), J. Sjöstrand (1982), A. Himonas (1986) in the setting of [analytic hypoellipticity for operators of principal type](#). An alternative proof has been given recently by J. Sjöstrand (2023).

Analytic regularity for operators with multiple characteristics : long tradition ... P. Albano – A. Bove (2013), A. Bove – M. Mughetti (2017, 2020), G. Chinni (2011), A. Bove – G. Chinni (2022), A. Parmeggiani (2022), W.-X Li – A. Parmeggiani (2013).

Reduction to hD_{x_1}

There exists a (vector-valued) FBI transform

$$T_h u(x) = \int e^{i\varphi(x,y)/h} a(x,y;h) \chi(y) u(y), \quad dy, \quad x \in \text{neigh}(0, \mathbb{C}^2),$$

such that for some $\delta > 0$,

$$hD_{x_1} T_h u(x) = T_h(Pu)(x) + \mathcal{O}(1)e^{(\Phi(x)-\delta)/h}, \quad x \in \text{neigh}(0, \mathbb{C}^2).$$

Here φ satisfies the [complex eikonal equation](#)

$$\varphi'_{x_1}(x,y) = q(y, -\varphi'_y(x,y)), \quad (x,y) \in \text{neigh}(0, \mathbb{C}^2) \times \text{neigh}(y_0, \mathbb{C}^2),$$

$$-\varphi'_y(0, y_0) = \eta_0, \quad \text{Im } \varphi''_{yy}(0, y_0) > 0, \quad \det \varphi''_{xy}(0, y_0) \neq 0.$$

This a well known consequence of [analytic WKB](#) in the scalar case (J. Sjöstrand (1982)), and it also works for [principally scalar systems](#).

Subharmonic minorants

It follows that $U(x; h) = T_h u(x)$ is essentially independent of x_1 ,

$$hD_{x_1} U(x; h) = \mathcal{O}(1)e^{(\Phi(x)-\delta)/h}, \quad x \in \text{neigh}(0, \mathbb{C}^2),$$

and therefore

$$|U(x; h)| \leq \mathcal{O}_\varepsilon(1)e^{(\Psi_\eta(x_2)+\varepsilon)/h}, \quad |x_1| < \eta, \quad |x_2| < \eta.$$

Here

$$\Psi_\eta(x_2) = \inf_{|x_1| < \eta} \Phi(x_1, x_2)$$

need no longer be subharmonic \implies we get

$$|U(x; h)| \leq \mathcal{O}_\varepsilon(1)e^{(\tilde{\Psi}_\eta(x_2)+\varepsilon)/h}, \quad |x_1| < \eta, \quad |x_2| < \eta,$$

where $\tilde{\Psi}_\eta$ is the largest subharmonic minorant of Ψ_η . If we can show that

$$\tilde{\Psi}_\eta(0) < \Phi(0)$$

for all small $\eta > 0$, then $\rho \notin \text{WF}_{a,h}(u)$. (Idea of Kashiwara (1979).)

Choosing the weight I

The complex eikonal equation has many solutions, so how do we choose the right one?

Model case : $q(y, \eta) = \eta_1 + i\eta_2 + iy_1^2$, $(y_0, \eta_0) = (0, 0)$.

Proposition (long tradition ... A. Himonas 1986)

There exists a real analytic canonical transformation

$$\kappa : \text{neigh}((y_0, \eta_0), T^*U) \rightarrow \text{neigh}((0, 0), T^*\mathbb{R}^2), \quad \kappa(y_0, \eta_0) = (0, 0),$$

and a real analytic function a defined in a neighborhood of $(0, 0)$, with $a(0, 0) \neq 0$, such that

$$q \circ \kappa^{-1} = a(y, \eta)q_0(y, \eta), \quad q_0(y, \eta) := \eta_1 + i(\eta_2 + y_1g(y, \eta_2)),$$

where g is real valued real analytic satisfying $g(0) = 0$, $g'_{y_1}(0) \neq 0$, and $g'_{y_2}(0) = 0$.

Choosing the weight II

To work with the approximate model symbol in the proposition, we use a **scaling argument** (taking $a = 1$ for simplicity),

$$\tilde{q}_0(y, \eta) = \frac{1}{\mu^2} q_0(\mu y, \mu^2 \eta), \quad 0 < \mu \ll 1.$$

We have, for $0 \neq c \in \mathbb{R}$,

$$\tilde{q}_0(y, \eta) = \eta_1 + i\eta_2 + icy_1^2 + \mu \mathcal{O}((y, \eta)^2).$$

It follows that the **complex eikonal equation**

$$\begin{cases} \varphi'_{x_1}(x, y) = \tilde{q}_0(y, -\varphi'_y(x, y)), \\ \varphi|_{x_1=0} = \frac{i}{2}(x_2 - y_2)^2 + iy_1^2 \end{cases}$$

has a unique solution in a small fixed neighborhood of $(0, 0) \in \mathbb{C}_x^2 \times \mathbb{C}_y^2$,

$$\varphi(x, y) = \frac{i}{2}(x_2 - y_2 + ix_1)^2 + i(y_1 - x_1)^2 + \frac{ic}{3}(y_1^3 - (y_1 - x_1)^3) + \mu \mathcal{O}((x, y)^3).$$

Choosing the weight III

The corresponding weight is given by

$$\Phi(x) = \frac{1}{2}(\operatorname{Im} x_2 + \operatorname{Re} x_1)^2 + (\operatorname{Im} x_1)^2 - \frac{1}{3}c(\operatorname{Re} x_1)^3 + \mathcal{O}(|x_1|^4) + \mathcal{O}(\mu)|x|^3,$$

and in particular,

$$\Psi_\eta(x_2) = \inf_{|x_1| < \eta} \Phi(x) \leq f(x_2) + \mathcal{O}(|x_2|^4) + \mathcal{O}(\mu)|x_2|^3, \quad |x_2| < \eta,$$

where

$$f(\zeta) = \frac{c}{3} (\operatorname{Im} \zeta)^3$$

is **superharmonic** for $\operatorname{Im} \zeta < 0$, (for $c > 0$). The largest subharmonic minorant U of f in the disc $|\zeta| < 1$ satisfies therefore

$$U(0) \leq \frac{1}{\pi} \iint_{D(0,1)} U(\zeta) L(d\zeta) < \frac{1}{\pi} \iint_{D(0,1)} f(\zeta) L(d\zeta) = 0.$$

It follows then that

$$\tilde{\Psi}_\eta(0) < \Phi(0) = 0,$$

for all $\eta > 0$ and $\mu > 0$ small enough.

Choosing the weight IV

Associated to the phase function φ is the **complex canonical transformation**

$$\kappa_\varphi : T^*\mathbb{C}^2 \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in T^*\mathbb{C}^2,$$

which satisfies

$$\kappa_\varphi(\text{neigh}((0, 0), T^*\mathbb{R}^2)) = \Lambda_\Phi \subset T^*\mathbb{C}^2,$$

where

$$\Lambda_\Phi := \left\{ \left(x, \frac{2}{i} \partial_x \Phi(x) \right) ; x \in \text{neigh}(0, \mathbb{C}^2) \right\}.$$

We should incorporate the real canonical transformation giving the approximate model symbol into an FBI transform.

Choosing the weight V

Proposition

Let $\kappa : \text{neigh}((y_0, \eta_0), T^*U) \rightarrow \text{neigh}((0, 0), T^*\mathbb{R}^2)$, $\kappa(y_0, \eta_0) = (0, 0)$, be a real analytic canonical transformation. Then the composition $\kappa_\varphi \circ \kappa$ is of the form

$$\kappa_\varphi \circ \kappa = \kappa_\psi : T^*\mathbb{C}^2 \ni (y, -\psi'_y(x, y)) \mapsto (x, \psi'_x(x, y)) \in T^*\mathbb{C}^2,$$

where $\psi = \psi(x, y) \in \text{Hol}(\text{neigh}((0, y_0), \mathbb{C}^4))$ satisfies

$$-\psi'_y(0, y_0) = \eta_0, \quad \text{Im } \psi''_{yy}(0, y_0) > 0, \quad \det \psi''_{xy}(0, y_0) \neq 0.$$

J. Sjöstrand (1983). ([Positivity](#) of complex Lagrangian planes.)

Remark. We have

$$\kappa_\psi(\text{neigh}((y_0, \eta_0), T^*U)) = \kappa_\varphi(\text{neigh}((0, 0), T^*\mathbb{R}^2)) = \Lambda_\Phi,$$

so the weight is **unchanged**.

One word about Step II

We have

$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x, hD_x)) u = 0 \quad \text{in } U,$$

where

$$\|u\|_{L^2(U)} \leq \mathcal{O}(1).$$

If $x_0 \in U$ satisfies the assumptions of the Theorem, then [Step I](#) gives :

$$\text{WF}_{a,h}(u) \cap q^{-1}(0) \cap \pi^{-1}(x_0) = \emptyset.$$

Here we recall that

$$q(x, \xi) = \sum_{|\alpha| \leq 2} a_\alpha(x) \xi^\alpha$$

is [classically elliptic](#),

$$\left| \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \right| \geq \frac{1}{C} |\xi|^2, \quad (x, \xi) \in T^*U.$$

Proposition

Let $(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x, hD_x)) u = 0$ in U , $x_0 \in U$, and assume that

$$\text{WF}_{a,h}(u) \cap q^{-1}(0) \cap \pi^{-1}(x_0) = \emptyset,$$

where q is classically elliptic. Then there exists a neighborhood Ω of x_0 and $C_0, c_0 > 0$ such that

$$\left| \partial^\beta u(x; h) \right| \leq C_0 (|\beta| C_0)^{|\beta|} e^{-c_0/h}, \quad x \in \Omega, \quad \beta \in \mathbb{N}^n.$$

This result is closely related to A. Martinez (2002) and can also be obtained as a consequence of the theory of [global FBI transforms](#) and [global exponentially weighted spaces](#) developed by J. Galkowski – M. Zworski (2021, 2022), allowing exponential weights which are not compactly supported in ξ .

Based on B. Helffer – J. Sjöstrand (1986), J. Sjöstrand (1996).

Back to the chiral model of TBG

We have

$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x)) u = 0,$$

where

$$q(x, \xi) = (2\bar{\zeta})^2 - U(z)U(-z), \quad z = x_1 + ix_2, \quad \zeta = \frac{1}{2}(\xi_1 - i\xi_2).$$

Symplectic structure on $T^*\mathbb{R}^2$:

$$\sigma = d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2 = 2\operatorname{Re}(d\zeta \wedge dz) = d\zeta \wedge dz + d\bar{\zeta} \wedge d\bar{z}.$$

Poisson bracket :

$$\{a, b\} = a'_\zeta b'_z - b'_\zeta a'_z + a'_{\bar{\zeta}} b'_{\bar{z}} - b'_{\bar{\zeta}} a'_{\bar{z}}.$$

Exponential decay of solutions near x_0 is guaranteed by $q(x_0, \xi) = 0 \implies$

$$\{q, \bar{q}\}(x_0, \xi) = 0, \quad \{q, \{q, \bar{q}\}\}(x_0, \xi) \neq 0, \quad H_{\operatorname{Re} q}(x_0, \xi) \not\parallel H_{\operatorname{Im} q}(x_0, \xi).$$

We have

$$q = 0 \iff 2\bar{\zeta} = \pm \sqrt{U(z)U(-z)},$$

and

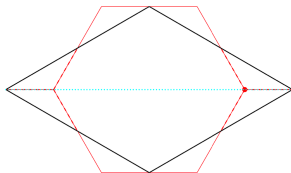
$$\{q, \bar{q}\}|_{q^{-1}(0)} = \pm 8i \operatorname{Im} \left((\overline{U(z)U(-z)})^{\frac{1}{2}} \partial_z (U(z)U(-z)) \right).$$

Let

$$H := \bigcup_{\pm} \bigcup_{k=0}^2 \pm(1 + \omega^k [0, \frac{1}{2}])z_S + \Lambda$$

be the **hexagon** spanned by the **stacking points** $\pm z_S + \Lambda$, $z_S = i/\sqrt{3}$, $\omega z_S \equiv z_S \pmod{\Lambda}$. An elementary computation shows that

$$dq(\rho) \neq 0, \quad \{q, \bar{q}\}(\rho) = 0, \quad \rho \in \pi^{-1}(H) \cap q^{-1}(0).$$



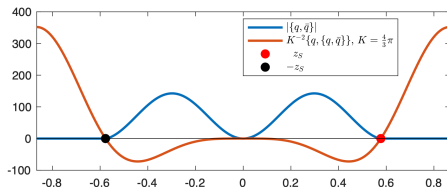
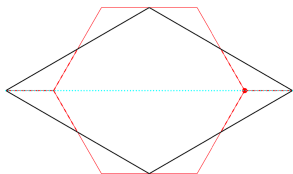
The second bracket

We have for $\rho \in q^{-1}(0) \cap \pi^{-1}(it)$, $it \in \pm z_S(1, 3/2]$,

$$\{q, \{q, \bar{q}\}\}(\rho) = -16V(\partial_z \partial_{\bar{z}} V + \overline{\partial_z^2 V}) + 8((\partial_z V)^2 - \partial_{\bar{z}} V \overline{\partial_z V}).$$

Here $V(z) = U(z)U(-z)$. It turns out that this expression can also be understood and we get

$$\{q, \{q, \bar{q}\}\}(\rho) = \frac{128}{9} \pi^2 (c-1)^2 (2c+1)(2c-9) \neq 0, \quad c := \cos(2\pi\sqrt{3}t/3).$$



Conclusion : $\{q, \{q, \bar{q}\}\}(\rho) \neq 0$ for $\rho \in q^{-1}(0) \cap \pi^{-1}(z)$, for z along the open edges of the hexagon \implies the theorem applies there.

What about the corners?

We have

$$q^{-1}(0) \cap \pi^{-1}(\pm z_S) = \{(\pm z_S, 0)\}, \quad dq(\pm z_S, 0) \neq 0,$$

and

$$\{q, \{q, \bar{q}\}\}(\pm z_S, 0) = 0,$$

with the first **non-vanishing bracket** given by

$$\{q, \{q, \{q, \{q, \bar{q}\}\}\}\}(\pm z_S, 0) = H_q^4 \bar{q}(\pm z_S, 0) \neq 0.$$

We have

$$q(z_S + z, \zeta) = 4\bar{\zeta}^2 + ia\bar{z} - bz^2 + \mathcal{O}(|z|^3), \quad a, b > 0.$$

Z. Tao – M. Zworski were recently able to treat the case of corners, by means of a direct analysis of the **complex eikonal equation**

$$\partial_{z_1} \varphi(z, w, v) = 4(\partial_v \varphi(z, w, v))^2 + iav - bw^2 + \mathcal{O}((v, w)^3), \quad z \in \mathbb{C}^2, \quad w, v \in \mathbb{C}.$$

Explicit detailed analysis of the eikonal equation shows that

$$\Psi(z_2) = \inf_{z_1} \Phi(z)$$

is of the form

$$\Psi(z_2) = -\frac{1}{3}\text{Im}(z_2^3) + |z_2|^2 \text{Im}(z_2^3) + \mathcal{O}(|z_2|^6).$$

Here the **largest subharmonic minorant** U of $|\zeta|^2 \text{Im}(\zeta^3)$ in the unit disk satisfies

$$U(0) < 0,$$

and hence we can proceed as before.

Exponential decay near the whole hexagon

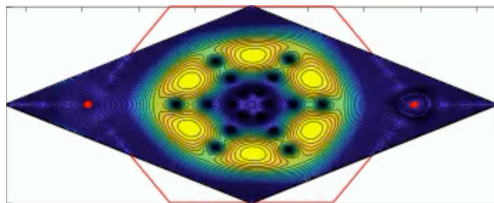
Theorem (Z. Tao – M. Zworski – M. H. (2023))

Assume that

$$(D(\alpha) + k)u = 0, \quad u \in H^1(\mathbb{C}/\Gamma; \mathbb{C}^2), \quad \|u\|_{L^2(\mathbb{C}/\Gamma; \mathbb{C}^2)} = 1.$$

Then there exists an α -independent open neighborhood Ω of the hexagon spanned by the stacking points and $C_0, c_0 > 0$ such that

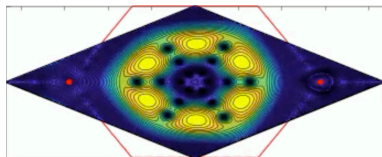
$$|u(z; \alpha)| \leq C_0 e^{-c_0 \alpha}, \quad z \in \Omega, \quad \alpha \geq 1.$$



What about the center of the hexagon ?

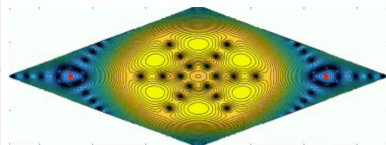
The origin $(x, \xi) = (0, 0)$ (the center of the hexagon) is a **doubly characteristic point** for q ,

$$q(0, 0) = 0, \quad dq(0, 0) = 0.$$



chiral model

$$(q(x, hD_x) + hR(x))u = 0$$



scalar model

$$q(x, hD_x)u = 0$$

Lower order terms do seem to matter in this case!



THANK YOU VERY MUCH FOR YOUR ATTENTION!

