Left-invariant involutive structures on compact Lie groups

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If $\mathcal{V}$ is such that $\mathcal{V} \oplus \overline{\mathcal{V}} = T_{\mathbb{C}}\Omega$, then $\mathcal{V}$ defines a compelx structure, $\mathrm{d}' = \overline{\partial}$, and $H^*(\Omega; \mathcal{V})$ corresponds to the Dolbeault cohomology.

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Example 3

If $\mathcal{V}$ is such that $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$, then $\mathcal{V}$ defines an abstract CR structure, $\mathrm{d}' = \overline{\partial}_b$, and $H^*(\Omega; \mathcal{V})$ corresponds to the tangential Dolbeault cohomology.

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Example 4

If $\mathcal{V}$ is such that $\mathcal{V} = \overline{\mathcal{V}}$, then $\mathcal{V}$ defines a real foliation of Ω, $\mathrm{d}' = \mathrm{d}$ "on the leaves", and $H^*(\Omega; \mathcal{V})$ corresponds to De Rham cohomology of the leaves.

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- What are the topological or geometrical conditions on Ω or $\mathcal{V}$ so that $H^q(\Omega; \mathcal{V}) = 0$ or $\dim H^q(\Omega; \mathcal{V}) < \infty$ or $H^q(\Omega; \mathcal{V})$ is a Hausdorff space?

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- Given $x \in \Omega$, what are the topological or geometrical conditions so that there is an open neighborhood $U \subset \Omega$ of x so that $H^q(U; \mathcal{V}) = 0$ or $\dim H^q(U; \mathcal{V}) < \infty$ or $H^q(U; \mathcal{V})$ is a Hausdorff space?

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Left-invariant involutive structures $\mathcal{V} \subset T_{\mathbb{C}}G$ are in a one-to-one correspondence with subalgebras $\mathfrak{v} \doteq \mathcal{V}_e \subset \mathfrak{g}_{\mathbb{C}}$.

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Question

How do we use algebraic properties of $\mathfrak{v}$ to better understand the analytic properties of $\mathcal{V}$ and $(C^\infty(G; \underline{\Lambda}^*), \mathrm{d}')$?

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- $(C_L^\infty(G;\underline{\Lambda}^*), \mathrm{d}')$ and $H_L^q(G;\mathfrak{v})$ depend only on $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{v}$;
- algebraic construction via Chevalley-Eilenberg complex:

$$(C^*(\mathfrak{v};\mathbb{C}), \mathrm{d}) \cong (C_L^\infty(G;\underline{\Lambda}^*), \mathrm{d}')$$

and

$$H^q(\mathfrak{g}_\mathbb{C};\mathfrak{v}) \cong H_L^q(G;\mathfrak{v}).$$

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Fact

The inclusion $C_L^\infty(G; \underline{\Lambda}^q) \subset C^\infty(G; \underline{\Lambda}^q)$ induces a homomorphism

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G compact $\implies \varphi$ injective.

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- Semisimple Elliptic structures; Araújo (2019) [6].

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- $\dim H^j(SU(2); \mathfrak{v}) = \infty$ for $j = 0, 1$.
- The map φ cannot be an isomorphism!

Left-invariant CR structures of maximal rank

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By the classification theorem of Charbonnel-Kalgui [7], there is a maximal torus $T \subset G$ such that $\mathcal{W}_t \doteq \mathcal{V}_t \cap \mathbb{C}\,\mathrm{T}_t\,T$ defines a bi-invariant CR structure $\mathcal{W} = \bigcup_{t \in T} \mathcal{W}_t$ on T.

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Definition

We call $\mathcal{W}$ the toric component of $\mathcal{V}$ (relative to the maximal torus T). We usually denote the toric component $\mathcal{W}$ by its corresponding Lie algebra $\mathfrak{m} \subset \mathbb{C}\mathfrak{t}$ with $\mathfrak{t}$ the Lie algebra of T.

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We say that $\mathcal{W}$ (or $\mathfrak{m}$) satisfies the divisor condition **(DC)** if there exist a basis $\{L_1, \ldots, L_n\}$ for $\mathcal{W}$ (or $\mathfrak{m}$) and constants $C, \rho > 0$ such that

$$\max_j |\widehat{L_j}(\xi)| \geq C(1 + |\xi|)^{-\rho}, \quad \forall \xi \in \mathbb{Z}^N,$$

with $\hat{L}_j$ being the symbol of the vector field L_j and N the dimension of T.

Theorem (Jacobowitz and J. (2023) [5])

Let G be a connected, odd-dimensional, and compact Lie group endowed with a left-invariant Levi-flat CR structure $\mathcal{V}$ of maximal rank. Suppose that $\mathcal{W}$, the toric part of $\mathcal{V}$, satisfies the **(DC)** condition, then there exist an isomorphism $H^q(G; \mathcal{V}) \cong H^q(T; \mathcal{W})$.

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Consequence

Under the same conditions, the map φ is an isomorphism.

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Theorem (J.) [8]

Suppose that G is a semisimple Lie group, and let $\mathfrak{v} \subset \mathfrak{g}_{\mathbb{C}}$ be elliptic. Let $G_{\mathbb{C}}$ be the universal complexification of G. If $V = \exp_{G_{\mathbb{C}}}(\mathfrak{v}) \subset G_{\mathbb{C}}$ is closed, then every cohomology class in $H^q(G;\mathfrak{v})$ has a left-invariant representative.
Moreover, the inclusion of the left-invariant complex into the usual one induces an isomorphism in cohomology:

$$\varphi : H^q(\mathfrak{g}_{\mathbb{C}};\mathfrak{v}) \to H^q(G;\mathfrak{v}).$$

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- to show equality we use spectral sequences.

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- $E_2^{j,k} \cong H^j(\mathfrak{k};\mathbb{C}) \oplus H^k(\mathfrak{v},\mathfrak{k},\mathbb{C})$ the second page.

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- $(\tilde{E}_r, \tilde{\mathrm{d}}_r)$: Leray spectral sequence with

$$\tilde{E}_2^{r,s} = H^r(\Omega; \mathcal{H}^s)$$

with limit term $H^*(G; \mathcal{S})$.

The Second Page

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Questions?