Recent results on the norm of localization operators

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Table of Contents

Short-time Fourier transform

Definition

Given $x, \omega \in \mathbb{R}$, the corresponding *translation*, *modulation* and *time-frequency shift operators* are defined as

$$T_x f(t) = f(t-x), \quad M_\omega f(t) = e^{2\pi i \omega t} f(t), \quad \pi(x,\omega) = M_\omega T_x, \quad t \in \mathbb{R}.$$

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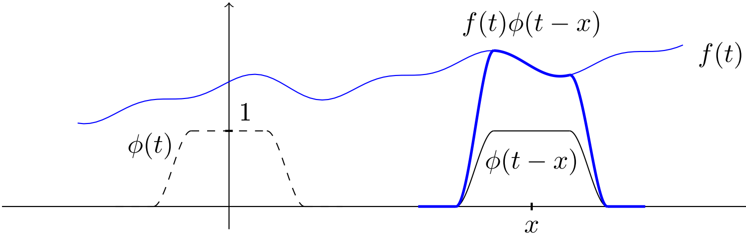
Short-time Fourier transform

The **short-time Fourier transform (Short-time Fourier transform)** with **window** $\phi \in L^2(\mathbb{R})$ of the function $f \in L^2(\mathbb{R})$ is defined as:

$$\mathcal{V}_\phi f(x, \omega) = \langle f, \pi(x, \omega)\phi \rangle = \mathscr{F}(f\overline{\phi(\cdot - x)})(\omega).$$

Short-time Fourier transform

Explaination of the definition and why the Gaussian window is “optimal”



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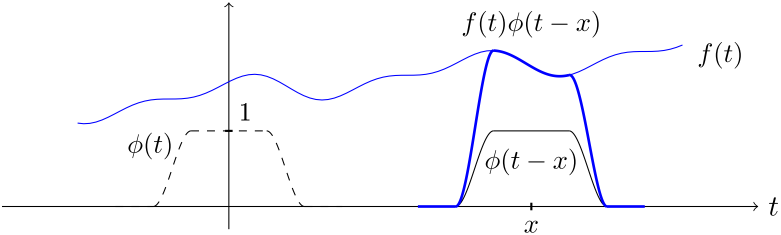

In order to have a good resolution for the STFT, the window function has to be well localized both in time and frequency. According to Heisenberg's uncertainty principle, Gaussian functions are optimal in this sense. Therefore, from now on, we fix the window to be a L^2-normalized Gaussian:

$$\varphi(t) = 2^{1/4} e^{-\pi t^2}, \quad t \in \mathbb{R}.$$

Table of Contents

Time-frequency localization operators

Where do localization operators come from?

With the particular choice of an L^2-normalized window, the STFT becomes an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$. Therefore, given $f \in L^2(\mathbb{R})$ with $\|f\|_2$, we have $\|\mathcal{V}_\varphi f\|_2 = 1$, so the quantity $|\mathcal{V}_\varphi f|^2$ can be seen as a time-frequency energy density of the function f.

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$$\int_\Omega |\mathcal{V}_\varphi f(x,\omega)|^2 \, dx d\omega$$

is the fraction of the energy of f “contained” in Ω. This can be written as

$$\int_\Omega |\mathcal{V}_\varphi f(x,\omega)|^2 \, dx d\omega = \langle \chi_\Omega \mathcal{V}_\varphi f, \mathcal{V}_\varphi f \rangle = \langle \mathcal{V}_\varphi^* \chi_\Omega \mathcal{V}_\varphi f, f \rangle.$$

Therefore, estimates on the norm of the operator $\mathcal{V}_\varphi^* \chi_\Omega \mathcal{V}_\varphi$ (i.e. the first eigenvalue) are important because they lead to estimates for the energy concentration of f.

Time-frequency localization operators

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Time-frequency localization operators

Given a function $F : \mathbb{R}^2 \to \mathbb{C}$, the **time-frequency localization operator** with **window** φ and **weight** F is defined as:

$$L_{F,\varphi} := \mathcal{V}_\varphi^* F \mathcal{V}_\varphi : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

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Localization operators are particular instances of pseudo-differential operators. For example, if we use Weyl quantization

$$(Op^w(a)f)(x) = \iint_{\mathbb{R}\times\mathbb{R}} e^{2\pi i(x-y)\omega} a\left(\frac{x+y}{2}, \omega\right) f(y)\, dy d\omega,$$

then we have that $L_{F,\varphi} = Op^w(a)$ with

$$a = F * \Phi, \quad \Phi(x,\omega) = 2e^{-2\pi(x^2+\omega^2)}.$$

Time-frequency localization operators

Properties

We present some properties of $L_{F,\varphi}$:

- if $F \in L^p(\mathbb{R}^2)$ with $p \in [1, +\infty]$ then $L_{F,\varphi}$ is bounded and $\|L_{F,\varphi}\| \leq \|F\|_p$;

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- if $F \in L^p(\mathbb{R}^2)$ then $L_{F,\varphi}$ is in the Schatten p-class S^p;
- for more general results about boundedness and compactness see, for example, [Cordero and Gröchenig 2003] or [Fernández and Galbis 2006];
- if F is radially symmetric then the eigenfunctions of $L_{F,\varphi}$ are Hermite functions.

Table of Contents

Optimal estimates
The problem

From now on we will present some estimates of the kind

$$\|L_{F,\varphi}\| \le C\|F\|_{\mathcal{B}} \tag{1}$$

where $\mathcal{B}$ is some Banach space. For example, we already mentioned that localization operators are bounded when $F \in L^p(\mathbb{R}^2)$ and that $\|L_{F,\varphi}\| \le \|F\|_p$, which is (1) with $\mathcal{B} = L^p(\mathbb{R}^2)$ and $C = 1$. However, here we are interested in **optimal** constant C in (1) and in finding optimal weight functions F, for which equality occurs in (1).

Optimal estimates

Dual version of Lieb's inequality

We start considering the optimal estimate for $F \in L^p(\mathbb{R}^2)$. This can be obtained through a duality argument using Lieb's inequality (from [Lieb 1978]), that is

$$\|\mathcal{V}_\varphi f\|_p^p \leq \frac{2}{p}\|f\|_2^p \tag{2}$$

for every $p \geq 2$ and every $f \in L^2(\mathbb{R})$, while optimal functions were obtained by Carlen in [Carlen 1991].

Optimal estimates

Dual version of Lieb's inequality

Combining previous results leads to the following theorem.

Lieb's inequality - dual form

Let $p \in (1, +\infty)$. Then, for every $F \in L^p(\mathbb{R}^2)$ it holds

$$\|L_{F,\varphi}\| \leq \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \|F\|_p, \tag{3}$$

with equality if and only if, for some $c \in \mathbb{C}$ and some $z_0 = (x_0, \omega_0) \in \mathbb{R}^2$,

$$F(z) = ce^{-\frac{\pi}{p-1}|z-z_0|^2}, \quad z = (x, \omega) \in \mathbb{R}^2. \tag{4}$$

Optimal estimates

Faber-Krahn inequality for the STFT

In [Nicola and Tilli 2022] the following theorem for the STFT was proved.

Faber-Krahn inequality for the STFT - Nicola and Tilli 2022

For every subset $\Omega \subset \mathbb{R}^2$ with finite measure and for every $f \in L^2(\mathbb{R}) \setminus \{0\}$, it holds

$$\frac{\int_\Omega |\mathcal{V}_\varphi f(x,\omega)|^2 \, dx d\omega}{\|f\|_2^2} \leq 1 - e^{-|\Omega|},$$

with equality if and only if Ω is a ball and

$$f(t) = ce^{2\pi i \omega_0 t} \varphi(t - x_0),$$

where $c \in \mathbb{C} \setminus \{0\}$ and (x_0, ω_0) is the center of Ω.

Optimal estimates

Faber-Krahn inequality for the STFT

An immediate corollary of previous theorem is the following estimate on the norm of localization operators whose weight function is a characteristic function.

Faber-Krahn inequality for the STFT - dual form

Let $\Omega \subset \mathbb{R}^2$ be finite and measurable. Then, letting $L_{\Omega,\varphi} = \mathcal{V}_\varphi^* \chi_\Omega \mathcal{V}_\varphi$, it holds:

$$\|L_{\Omega,\varphi}\| \leq 1 - e^{-|\Omega|},$$

with equality if and only if Ω is equivalent to a ball.

Optimal estimates

The case $F \in L^p \cap L^\infty$

Theorem - Nicola and Tilli 2023

Let $p \in (1, +\infty)$ and let $F \in L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $F \neq 0$.

1. If $\|F\|_p / \|F\|_\infty \leq (p-1)/p$ estimate (3) is still optimal.

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1. If $\|F\|_p/\|F\|_\infty \leq (p-1)/p$ estimate (3) is still optimal.
2. If $\|F\|_p/\|F\|_\infty > (p-1)/p$ then

$$\|L_{F,\varphi}\| \leq \left(1 - \frac{e^{(p-1)/p-(\|F\|_p/\|F\|_\infty)^p}}{p}\right)\|F\|_\infty,$$

with equality if and only if, for some $\theta \in \mathbb{R}$, some $z_0 = (x_0, \omega_0) \in \mathbb{R}^2$ and some $\lambda > \|F\|_\infty$,

$$F(z) = e^{i\theta} \min\{\lambda e^{-\frac{\pi}{p-1}|z-z_0|^2}, \|F\|_\infty\}, \quad z \in \mathbb{R}^2. \tag{5}$$

Optimal estimates

The case $F \in L^p \cap L^q$

Theorem - R. 2023

Let $p, q \in (1, +\infty)$ and let $F \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ with $F \neq 0$. Then, there exist two constants

$$r_1 = \left(\frac{q-1}{q}\right)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{p}{q}\right)^{\frac{1}{p}}, \quad r_2 = \left(\frac{p-1}{p}\right)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{p}{q}\right)^{\frac{1}{q}},$$

where $r_1 \leq r_2$, such that:

1. if $\|F\|_q/\|F\|_p \leq r_1$ or $\|F\|_q/\|F\|_p \geq r_2$, then estimate (3), with p and q respectively, is still optimal;

Optimal estimates

The case $F \in L^p \cap L^q$

Theorem - R. 2023

2. if $r_1 < \|F\|_q / \|F\|_p < r_2$ then

$$\|L_{F,\varphi}\| \le T - \lambda_1 T^p/p - \lambda_2 T^q/q, \tag{6}$$

where $\lambda_1, \lambda_2 > 0$ are uniquely determined by

$$p \int_0^{+\infty} t^{p-1} u(t)\, dt = A^p, \quad q \int_0^{+\infty} t^{q-1} u(t)\, dt = B^q,$$

with

$$u(t) = \max\{-\log(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}), 0\} \tag{7}$$

and $T > 0$ is such that $\lambda_1 T^{p-1} + \lambda_2 T^{q-1} = 1$. Finally, equality in (6) is achieved if and only if F is (up to translations) radially symmetric and has u as distribution function.

Optimal estimates

Idea of the proof

The first step is to rewrite the problem in the form of a constrained optimization problem:

Given $A, B > 0$ and $p, q \in (1, +\infty)$ find the best constant $C = C(p, q, A, B) > 0$ such that

$$\|L_{F,\varphi}\| \le C$$

for every F satisfying the following constraints:

$$\|F\|_p \le A, \quad \|F\|_q \le B. \tag{8}$$

Optimal estimates

Idea of the proof

Then, we need the following theorem from [Nicola and Tilli 2023].

Theorem - Nicola and Tilli 2023

Let $F \in L^p(\mathbb{R}^2)$ with $p \in [1, +\infty)$ and let $\mu(t) = |\{|F| > t\}|$ the distribution function of $|F|$. Then, it holds

$$\|L_{F,\varphi}\| \leq \int_0^{+\infty} (1 - e^{-\mu(t)})\, dt, \tag{9}$$

with equality if and only if F is (up to translations) radially symmetric.

Since this estimate is sharp, we should seek for sharp upper bounds for the right-hand side.

Optimal estimates

Idea of the proof

The corresponding variational problem is the following

$$\sup_{v\in\mathcal{C}} \int_0^{+\infty} (1 - e^{-v(t)})\, dt,$$

where $\mathcal{C}$ is the set of non-increasing functions $v : (0, +\infty) \to [0, +\infty)$ that satisfy

$$p \int_0^{+\infty} t^{p-1} v(t)\, dt \leq A^p, \quad q \int_0^{+\infty} t^{q-1} v(t)\, dt \leq B^q. \tag{10}$$

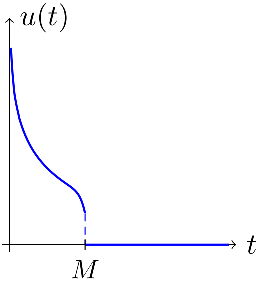
Optimal estimates

Idea of the proof

Once existence of a solution is proved (Helly's selection theorem), one can show that optimal functions are of the kind

$$u(t) = \begin{cases} -\log\left(\lambda_1 t^{p-1} + \lambda_2 t^{q-1}\right), & t \in (0, M) \\ 0, & t \in (M, +\infty) \end{cases}$$

for some $M > 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.



Example of optimal function

Optimal estimates

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Once that explicit expression of optimal functions is known, one can show that these achieve equality in the constraints, that the multipliers λ_1 and λ_2 are both positive and that the extremal functions are indeed continuous.

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Once that explicit expression of optimal functions is known, one can show that these achieve equality in the constraints, that the multipliers λ_1 and λ_2 are both positive and that the extremal functions are indeed continuous.

Lastly, one has to prove that multipliers are unique, which means that the system

$$\begin{cases} f(\lambda_1, \lambda_2) := p \int_0^T t^{p-1} u(t; \lambda_1, \lambda_2) \, dt = A^p \\ g(\lambda_1, \lambda_2) := q \int_0^T t^{q-1} u(t; \lambda_1, \lambda_2) \, dt = B^q \end{cases}$$

has a unique solution or, equivalently, that the level sets $\{f = A^p\}$ and $\{g = B^q\}$ intersect in only a point.

Optimal estimates

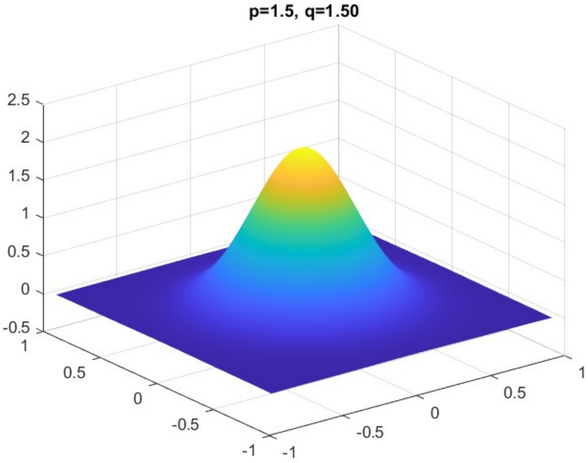
Idea of the proof

Thanks to the implicit function theorem one can show that these level sets can be seen as the graph of two functions, ϕ and γ, respectively. The proof is complete thanks to the following facts:

- the condition $r_1 < B/A < r_2$ is equivalent to the fact $\phi - \gamma$ changes sign in its domain;
- whenever ϕ and γ intersect, $\phi' < \gamma'$.

Optimal estimates

Optimal weights for $A = B = 1$, $p = 1.5$, q varies from 1.5 to 40



An immediate corollary: Lieb's inequality for $L^p + L^q$

Given $f \in L^2(\mathbb{R})$ and $p, q \in (1, +\infty)$ one has

$$\| |\mathcal{V}_\varphi f|^2 \|_{L^p + L^q} = \max_{\|F\|_{(L^p+L^q)'} \le 1} |\langle F, |\mathcal{V}_\varphi|^2 \rangle|$$

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so, using the previous theorem, we obtain an optimal estimate and the characterization of those f that achieve equality.

Table of Contents

Possible research directions

The research on such optimal estimate is very active (see, for example, [Kulikov 2022] or [Frank 2023]). Here are some possible directions for further research:

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- estimates for more general spaces (e.g. modulation spaces);
- estimates for other operator norms (e.g. Hilbert-Schmidt norm);
- quantitative version of the estimate;
- estimates for other types of localization operators (e.g. wavelet localization operators).

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Thank you for the attention

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