Ultradifferentiable classes of entire functions

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Introduction - I

(∗) Ultradifferentiable functions: Sub-classes of smooth functions such that the growth of $f^{(p)}$, $p \in \mathbb{N}$, is controlled/measured in terms of a weight.

(∗) "Classically" two approaches:
(i) weight sequence $M = (M_p)_{p\in\mathbb{N}}$ (e.g. S. Mandelbrojt; H. Cartan; H. Komatsu; L. Hörmander) or a
(ii) weight function $\omega : [0, +\infty) \to [0, +\infty)$ (e.g. A. Beurling; G. Björck; D. Vogt; H.-J. Petzsche; R. Braun, R. Meise, B. A. Taylor).

(∗) The weight sequence case has been introduced first.

(∗) In general both settings are mutually distinct (see Bonet/Meise/Melikhov '07; Rainer/S. '14).

Introduction - II

(∗) Growth and regularity assumptions on M and ω are required.

(∗) Conditions on weights imply, or even characterize, (desired) properties for the corresponding function classes.

(∗) From now on we focus on the weight sequence case.

(∗) Analogous definitions/results/constructions are expected for ω, too. - Open problem!!

Introduction - III

(∗) (One-dimensional case) Let $U \subseteq \mathbb{R}$ be open. For each compact $K \subset\subset U$, the set

$$\left\{ \frac{f^{(p)}(x)}{h^p M_p} \;:\; p \in \mathbb{N},\; x \in K \right\},$$

is required to be bounded.

(∗) Roumieu-type $\mathcal{E}_{\{M\}}$: boundedness for some $h > 0$
Beurling-type $\mathcal{E}_{(M)}$: boundedness for all $h > 0$

(∗) We can define such spaces for $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ arbitrary.

(∗) Usually, M is assumed to be "increasing fast".

Introduction - IV

(∗) What is fast? Set $m_p := \frac{M_p}{p!}$, then it is (often) standard to assume

$$\liminf_{p\to+\infty}(m_p)^{1/p} > 0$$

for the Roumieu-case and

$$\lim_{p\to+\infty}(m_p)^{1/p} = +\infty$$

for the Beurling-case.

(∗) Crucial to ensure that the real-analytic functions (R.-type with $M_p = p!$) are contained in $\mathcal{E}_{\{M\}}$ resp. in $\mathcal{E}_{(M)}$.

(∗) "Normally" $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ are supposed to be lying between known/important function classes.

Introduction - V

(∗) Note: In the literature the sequence $m := (m_p)_{p\in\mathbb{N}}$ is sometimes denoted by M.

(∗) But m and M must not be mixed!

(∗) Our results (also) illustrate the difference/growth gap between M and m.

Our aim(s)

(∗) Study $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ when M is violating standard growth requirements - "small sequences".

(∗) What are the differences between such small classes and spaces defined in terms of standard sequences?

(∗) For which applications can such "exotic classes" be useful?

(∗) Can we transfer known results from the standard setting to small spaces?

(∗) Can one construct from a given standard sequence "canonically" a small one (and vice versa)?

Our aim(s) - comments I

(∗) Main motivation: connection between classical/fast growing and exotic/small sequences.

(∗) Dual sequences: Introduced in J. Jiménez-Garrido's PhD-thesis ('18) for the study of certain growth indices for sequences. A different story...

(∗) Very few literature concerning small classes is available.

(∗) To the best of our knowledge we have only found works by M. Markin (approx. 2000)...

Our aim(s) - comments II

(∗) Markin has considered small Gevrey sequences:
$G^s := (p!^s)_{p\in\mathbb{N}}$ - equivalently $(p^{ps})_{p\in\mathbb{N}}$ - with $0 \leq s < 1$.
Compare: "normally" one has $s \geq 1$.

(∗) Given a Hilbert space H and a normal (unbounded) operator A on H, then consider the evolution equation

$$y'(t) = Ay(t),$$

and ask: Is it possible to detect boundedness of A in terms of regularity of all (weak) solutions $y : [0, +\infty) \to H$?

(∗) Markin has shown: If each weak solution y (notion weak w.r.t. the adjoint A^*) belongs to some small Gevrey class, then A is bounded.

Weight sequences - I

(∗) Let $M = (M_p)_p \in \mathbb{R}_{>0}^{\mathbb{N}}$ and set $m = (m_p)_p$ with $m_p := \frac{M_p}{p!}$.

(∗) M is called normalized, if $1 = M_0 \leq M_1$.

(∗) M is called log-convex, if

$$\forall\, p \in \mathbb{N}_{>0}:\ M_p^2 \leq M_{p-1} M_{p+1}.$$

(∗) We introduce the set

$$\mathcal{LC} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}}:\ M \text{ is norm., l.c., } \lim_{p\to+\infty} (M_p)^{1/p} = +\infty\}.$$

Weight sequences - II

(∗) Given $M = (M_p)_{p\in\mathbb{N}}$ and $N = (N_p)_{p\in\mathbb{N}}$ we write $M \le N$ if $M_p \le N_p$ for all $p \in \mathbb{N}$ and $M \preceq N$ if

$$\sup_{p\in\mathbb{N}_{>0}} \left(\frac{M_p}{N_p}\right)^{1/p} < +\infty.$$

(∗) M and N are called equivalent, if $M \preceq N$ and $N \preceq M$.

(∗) Above one can replace M and N simultaneously by m and n.

Ultradifferentiable classes - I

(∗) Let $\mathcal{E}$ denote the class of smooth functions. Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$, $U \subseteq \mathbb{R}^d$ be non-empty open.

(∗) Define the (local) classes of Roumieu-type by

$$\mathcal{E}_{\{M\}}(U) := \{f \in \mathcal{E}(U) : \ \forall\, K \subset\subset U\ \exists\, h > 0 : \ \|f\|_{M,K,h} < +\infty\},$$

(∗) and the Beurling-type by

$$\mathcal{E}_{(M)}(U) := \{f \in \mathcal{E}(U) : \ \forall\, K \subset\subset U\ \forall\, h > 0 : \ \|f\|_{M,K,h} < +\infty\},$$

(∗) where we set

$$\|f\|_{M,K,h} := \sup_{\alpha \in \mathbb{N}^d, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

Ultradifferentiable classes - II

(∗) We write $\mathcal{E}_{[M]}$ if we mean either $\mathcal{E}_{\{M\}}$ or $\mathcal{E}_{(M)}$.

(∗) We omit writing the open set U if we do not want to specify the set where the functions are defined.

(∗) Analogously one can define classes with values in Hilbert or even Banach spaces H (for simplicity here we assume $U \subseteq \mathbb{R}$):

$$\|f\|_{M,K,h} := \sup_{p\in\mathbb{N},x\in K} \frac{\|f^{(p)}(x)\|_H}{h^p M_p},$$

i.e. the absolute value of $f^{(p)}(x)$ is replaced by the norm $\|\cdot\|_H$.

(∗) We write $\mathcal{E}_{[M]}(U,H)$ for this vector-valued classes and omit H if functions are scalar-valued.

Ultradifferentiable classes - III

(∗) Similarly for holomorphic/entire functions - write $\mathcal{H}(\mathbb{C},H)$.

(∗) Let $U\subseteq\mathbb{R}$ be open and connected. Then $\mathcal{E}_{(G^1)}(U,H)$ can be identified with $\mathcal{H}(\mathbb{C},H)$. The isomorphism $\cong$ (as Fréchet spaces) is given by

$$E:\mathcal{E}_{(G^1)}(U,H)\rightarrow\mathcal{H}(\mathbb{C},H),\quad f\mapsto E(f):=\sum_{k=0}^{+\infty}\frac{f^{(k)}(x_0)}{k!}z^k,$$

where x_0 is any fixed point in U.

(∗) The inverse is given by restriction to U.

Small sequences

Lemma

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ *be given.*

(i) *If* $\lim_{p\to+\infty}(m_p)^{1/p} = 0$, *then* $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{(G^1)}(\cong \mathcal{H}(\mathbb{C}))$ *with continuous inclusion.*

(ii) *Let* M *be log-convex and normalized. Assume that*

$$\mathcal{E}_{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}_{(G^1)}(\mathbb{R})(\cong \mathcal{H}(\mathbb{C})),$$

holds (as sets), then $\lim_{p\to+\infty}(m_p)^{1/p} = 0$ *follows.*

Conjugate sequence - I

(∗) Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ - define the conjugate sequence $M^* = (M_p^*)_{p\in\mathbb{N}}$ by

$$M_p^* := \frac{p!}{M_p} = \frac{1}{m_p}, \quad p \in \mathbb{N},$$

i.e. $M^* := m^{-1}$.

(∗) There is a one-to-one correspondence between M and M^* and $M^{**} = M$ holds.

(∗) (Known) growth properties for M can be expressed in terms of M^* - and vice versa...

Conjugate sequence - II

- (∗) $M \preceq N$ if and only if $N^* \preceq M^*$ and so $M \approx N$ if and only if $M^* \approx N^*$.
- (∗) $\lim_{p\to+\infty}(M_p^*)^{1/p} = +\infty$ if and only if $\lim_{p\to+\infty}(m_p)^{1/p} = 0$.
- (∗) M^* is log-convex if and only if m is log-concave (non-standard!), i.e.

$$\forall\, p \in \mathbb{N}_{>0}: \quad m_p^2 \geq m_{p-1}m_{p+1}.$$

Conjugate sequence - III

Lemma

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be given with $1 = M_0 \geq M_1$ and let M^* be the conjugate sequence.

(a) $M^* \in \mathcal{LC}$ *if and only if* m *is log-concave and* $\lim_{p\to+\infty}(m_p)^{1/p} = 0$.

(b) $M^* \in \mathcal{LC}$ *implies* $\mathcal{E}_{\{M\}} \subsetneq \mathcal{E}_{(G^1)}$ *with strict inclusion.*

(c) *If in addition* M *is log-convex with* $1 = M_0 = M_1$*, then* $\mathcal{E}_{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}_{(G^1)}(\mathbb{R})$ *gives* $\lim_{p\to+\infty}(M_p^*)^{1/p} = +\infty$.

Conjugate sequence - IV

Lemma

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be given. Then the following are equivalent:

(*i*) *We have $M \preceq M^*$.*

(*ii*) *We have $M \preceq G^{1/2}$.*

(*iii*) *We have $G^{1/2} \preceq M^*$.*

Analogously, if $M^* \preceq M$ resp. if $\preceq$ is replaced by $\leq$. Thus:

($*$) *$M \approx M^*$ if and only if $M \approx G^{1/2}$ and*

($*$) *$M = M^*$ if and only if $M = G^{1/2} = M^*$.*

($*$) *In particular, $G^{1/2} = (G^{1/2})^*$ holds true.*

(Markin's) Example - small Gevrey sequences

(∗) Let $M \equiv G^s$ for $0 \leq s < 1$ and $G^s \in \mathcal{LC}$.

(∗) So $m \equiv G^{s-1}$ with $-1 \leq s-1 < 0$ (negative Gevrey-index!).

(∗) We have $\lim_{p \to +\infty}(m_p)^{1/p} = 0$ and m is log-concave.

(∗) $M^* \equiv G^{1-s}$ and so clearly $M^* \in \mathcal{LC}$, too.

Weighted entire spaces - I

(∗) Let v be the weight function: $v : [0, +\infty) \to (0, +\infty)$ is radial and v is

- (−) continuous,
- (−) non-increasing and
- (−) rapidly decreasing, i.e. $\lim_{t \to +\infty} t^k v(t) = 0$ for all $k \geq 0$.

(∗) We call v normalized, when $v(t) = 1$ for all $t \in [0, 1]$ (w.l.o.g.).

(∗) Let H be a Hilbert space, we consider H-valued weighted spaces of entire functions:

$$\mathcal{H}^\infty_v(\mathbb{C}, H) := \{f \in \mathcal{H}(\mathbb{C}, H) : \|f\|_v := \sup_{z \in \mathbb{C}} \|f(z)\|_H v(|z|) < +\infty\}.$$

Weighted entire spaces - II

(∗) Let $\underline{\mathcal{V}} = (v_n)_{n \in \mathbb{N}_{>0}}$ be a non-increasing sequence of weights, i.e. $v_n \geq v_{n+1}$ for all n.
Define the (LB)-space

$$\mathcal{H}^\infty_{\underline{\mathcal{V}}}(\mathbb{C}, H) := \varinjlim_{n \to \infty} \mathcal{H}^\infty_{v_n}(\mathbb{C}, H).$$

(∗) If $\overline{\mathcal{V}} = (v_n)_{n \in \mathbb{N}_{>0}}$ is a non-decreasing sequence of weights, i.e. $v_n \leq v_{n+1}$ for all n, then define the Fréchet-space

$$\mathcal{H}^\infty_{\overline{\mathcal{V}}}(\mathbb{C}, H) := \varprojlim_{n \to \infty} \mathcal{H}^\infty_{v_n}(\mathbb{C}, H).$$

Weighted entire spaces - III

(∗) Let v be a weight function and $c > 0$. Set

$$v_c : t \mapsto v(ct), \qquad v^c : t \mapsto v(t)^c.$$

Each v_c and v^c is a weight.

(∗) Consider the dilatation-type system

$$\underline{\mathcal{V}}_{\mathfrak{c}} := (v_c)_{c \in \mathbb{N}_{>0}}, \qquad \overline{\mathcal{V}}_{\mathfrak{c}} := (v_{\frac{1}{c}})_{c \in \mathbb{N}_{>0}}.$$

(∗) Similarly, if $v \leq 1$, the exponential-type system

$$\underline{\mathcal{V}}^{\mathfrak{c}} := (v^c)_{c \in \mathbb{N}_{>0}}, \qquad \overline{\mathcal{V}}^{\mathfrak{c}} := (v^{\frac{1}{c}})_{c \in \mathbb{N}_{>0}}.$$

Weighted entire spaces - IV

(∗) Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be with $M_0 = 1$, such that M is log-conv. and $\lim_{p\to+\infty}(M_p)^{1/p} = +\infty$.

(∗) For such M consider the associated weight function $\omega_M : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by

$$\omega_M(t) := \sup_{j\in\mathbb{N}} \log\left(\frac{t^j}{M_j}\right) \quad \text{for } t \neq 0, \qquad \omega_M(0) := 0.$$

(∗) Let $c > 0$, then set

$$v_M(t) := \exp(-\omega_M(t)), \quad t \geq 0,$$

$$v_{M,c}(t) := \exp(-\omega_M(ct)), \; v_M^c(t) := \exp(-c\omega_M(t)).$$

Weighted entire spaces - V

(∗) Introduce $\underline{\mathcal{M}}_{\mathfrak{c}} := (v_{M,c})_{c\in\mathbb{N}_{>0}}$, $\overline{\mathcal{M}}_{\mathfrak{c}} := (v_{M,\frac{1}{c}})_{c\in\mathbb{N}_{>0}}$, $\underline{\mathcal{M}}^{\mathfrak{c}} := (v_M^c)_{c\in\mathbb{N}_{>0}}$ and $\overline{\mathcal{M}}^{\mathfrak{c}} := (v_M^{\frac{1}{c}})_{c\in\mathbb{N}_{>0}}$.

(∗) We write $\underline{\mathcal{M}^*}_{\mathfrak{c}}, \dots$ for the systems corresponding to M^*.

Main result - Markin for $M \cong G^s$, $0 \leq s < 1$

Theorem

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $M_0 = 1 \geq M_1$ such that $\lim_{p\to+\infty}(m_p)^{1/p} = 0$ and m is log-concave. Let $I \subseteq \mathbb{R}$ be an interval, then

$$E : \mathcal{E}_{\{M\}}(I, H) \to \mathcal{H}^{\infty}_{\underline{\mathcal{M}^*}_{\mathfrak{c}}}(\mathbb{C}, H), \quad f \mapsto E(f) := \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(z - x_0)^k$$

is an isomorphism (of locally convex spaces) for any fixed $x_0 \in I$. Moreover, with the same definition for E, also

$$E : \mathcal{E}_{(M)}(I, H) \to \mathcal{H}^{\infty}_{\overline{\mathcal{M}^*}_{\mathfrak{c}}}(\mathbb{C}, H)$$

is an isomorphism.

Comparison with Markin's statement

- (∗) Let $0 \leq \alpha < 1$ (fixed) and put $v(t) := e^{-t^{1/(1-\alpha)}}$.
- (∗) Markin has shown that the following mappings are isomorphisms (as l.c.v.s.):

$$E : \mathcal{E}_{\{G^\alpha\}}(I, H) \to \mathcal{H}^\infty_{\underline{\mathcal{V}}^c}(\mathbb{C}, H),$$

and

$$E : \mathcal{E}_{(G^\alpha)}(I, H) \to \mathcal{H}^\infty_{\overline{\mathcal{V}}^c}(\mathbb{C}, H).$$

- (∗) This follows by our result applied to G^α, $0 \leq \alpha < 1$, by $(G^\alpha)^* = G^{1-\alpha}$, by computing the corresponding associated function and finally comparing the dilatation- and exponential-type growth systems - possible for Gevrey-weights!

An application - "bad" M = "nice" M^*

Theorem

Let $M, N \in \mathbb{R}_{>0}^{\mathbb{N}}$ be given and assume that

- $(*)$ $1 = M_0 \geq M_1$ *and* $1 = N_0 \geq N_1$,
- $(*)$ $\lim_{p\to+\infty}(m_p)^{1/p} = \lim_{p\to+\infty}(n_p)^{1/p} = 0$,
- $(*)$ *both m and n are log-concave.*

Then the following are equivalent:

- (i) *We have $M \preceq N$.*
- (ii) *We have $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{N\}}$ with continuous inclusion.*
- (iii) *We have $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(N)}$ with continuous inclusion.*

Proof: Combine the previous main Theorem and the recent characterizations for inclusions for weighted entire spaces (S. '22) applied to the conjugates.

Intro - I

(∗) Let H be a Hilbert and space and A a normal (unbounded) operator H. Consider

$$y'(t) = Ay(t). \tag{1}$$

(∗) If A is a bounded operator on H, then each solution y of (1) is an entire function of exponential type.

(∗) M. Markin: There exists an unbounded normal operator A such that each (weak) solution of (1) is an entire function.

(∗) Thus, in order to detect boundedness for A, more precise regularity/growth restriction is required!

Intro - II

We generalize a first result from Markin:

Theorem

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be given and $I \subseteq \mathbb{R}$ a closed interval. Then a solution y of (1) belongs to $\mathcal{E}_{[M]}(I,H)$ if and only if $y(t) \in \mathcal{E}_{[M]}(A)$ for all $t \in I$. In this case one has $y^{(n)}(t) = A^n y(t)$ for all $t \in I$.

Here the R.-case is

$$\mathcal{E}_{\{M\}}(A) := \{f \in C^\infty(A) : \quad \exists\, C,h>0\; \forall\, n \in \mathbb{N} \quad \|A^n f\|_H \le C h^n M_n\},$$

with

$$C^\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n),$$

and $D(A^n)$ is the domain of A^n, the n-fold iteration of A (densely defined on H).

Intro - III

Markin has shown the following:

Lemma

Let $0 < \beta < +\infty$. If (as sets)

$$\bigcup_{0<\beta'<\beta} \mathcal{E}_{\{G^{\beta'}\}}(A) = \mathcal{E}_{(G^{\beta})}(A),$$

then the operator A is bounded.

Goal: Generalize this to more arbitrary families of small sequences.

Lemma looks strange for "common classes"...

But "common$=A=$differential operator"...unbounded!

Our results - crucial lemma

Lemma

Let $\mathfrak{F} \subseteq \mathcal{LC}$ be a family of sequences such that

$$\forall\, N \in \mathfrak{F}\, \exists\, M \in \mathfrak{F} : \quad \omega_M(2t) = O(\omega_N(t)) \text{ as } t \to +\infty. \tag{2}$$

Suppose there exists $\mathbf{a} = (a_j) \in \mathbb{R}_{>0}^{\mathbb{N}}$ *with:*

(*i*) $\lim_{j\to+\infty}(a_j)^{1/j} = 0$,

(*ii*) $\mathbf{a}$ *is a uniform bound for* $\mathfrak{F}$*, i.e.*

$$\forall\, N \in \mathfrak{F}\, \exists\, C > 0\, \forall\, j \in \mathbb{N} : \quad \frac{N_j}{j!} = n_j \leq C a_j.$$

If as sets $\bigcup_{N\in\mathfrak{F}} \mathcal{E}_{\{N\}}(A) = \mathcal{E}_{(G^1)}(A)$*, then A is bounded.*

Our results - technical construction I

Requirements for $\mathfrak{F}$?

Let $\mathfrak{F} := \{N^{(\beta)} \in \mathbb{R}_{>0}^{\mathbb{N}} : \beta > 0\}$ such that

(*i*) $N_0^{(\beta)} = 1$ for all $\beta > 0$,

(*ii*) $N^{(\beta_1)} \leq N^{(\beta_2)} \Leftrightarrow n^{(\beta_1)} \leq n^{(\beta_2)}$ for all $0 < \beta_1 \leq \beta_2$ (point-wise order),

(*iii*) $\lim_{j\to+\infty}(n_j^{(\beta)})^{1/j} = 0$ for each $\beta > 0$,

(*iv*) $j \mapsto (n_j^{(\beta)})^{1/j}$ is non-increasing for every $\beta > 0$,

(*v*) $\lim_{j\to+\infty}\left(\frac{N_j^{(\beta_2)}}{N_j^{(\beta_1)}}\right)^{1/j} = \lim_{j\to+\infty}\left(\frac{n_j^{(\beta_2)}}{n_j^{(\beta_1)}}\right)^{1/j} = +\infty$ for all $0 < \beta_1 < \beta_2$.

Our results - technical construction II

Proposition

Let $\mathfrak{F} := \{N^{(\beta)} \in \mathbb{R}_{>0}^{\mathbb{N}} : \beta > 0\}$ *have* $(i) - (v)$ *from before.*
Then there exists $\mathbf{a} = (a_j)_j \in \mathbb{R}_{>0}^{\mathbb{N}}$ *such that*

- $(*)$ $j \mapsto (a_j)^{1/j}$ *is non-increasing,*
- $(*)$ $(a_j)^{1/j} \to 0$ *as* $j \to +\infty$*, and*
- $(*)$ $\lim_{j\to+\infty} \left(\frac{a_j}{n_j^{(\beta)}}\right)^{1/j} = +\infty$ *for all* $\beta > 0$*.*

In addition, $\mathfrak{F}$ *satisfies* (2)*.*

Our results - comments on the proof

(∗) The proof of the crucial lemma before requires another technical preparation.

(∗) One proceeds by contradiction.

(∗) One uses an alternative description for $\mathcal{E}_{\{M\}}(A)$ involving the associated function and the spectral measure associated with A (due to Gorbachuk and Knyazyuk '89).

Main result - I

Theorem

Let $\mathbf{a} = (a_j)_j$ *such that* $a_j^{1/j} \to 0$ *and* $\mathfrak{F}$ *as before. Assume that for any weak solution* y *of* (1) *on* $[0, +\infty)$*, there is* $N \in \mathfrak{F}$ *such that* $y \in \mathcal{E}_{\{N\}}([0, +\infty), H)$.

Then the operator A *is bounded.*

Combining this with the crucial representation involving the conjugate sequence we obtain...

Main result - II

For $\mathfrak{F}$ consider:

($\mathfrak{F}_1$) $N \in \mathcal{LC}$ for all $N \in \mathfrak{F}$ and $1 = N_0 = N_1$,

($\mathfrak{F}_2$) $\mathfrak{F}$ has (2),

($\mathfrak{F}_3$) $\mathfrak{F}$ is uniformly bounded by some $\mathbf{a} = (a_j)_j$ with $(a_j)^{1/j} \to 0$, and

($\mathfrak{F}_4$) for all $N \in \mathfrak{F}$ we have that n is log-concave.

Theorem

Let $\mathfrak{F}$ satisfy $(\mathfrak{F}_1) - (\mathfrak{F}_4)$. Suppose that for every weak solution y of (1) there exist $N \in \mathfrak{F}$ and $C, k > 0$ such that y can be extended to an entire function with

$$\|y(z)\|_H \leq Ce^{\omega_{N^*}(k|z|)}.$$

Then A is already a bounded operator.

Dual - I

Goal: Find a natural construction for "exotic/non-standard/small" sequences.

- (∗) Immediate: start with "nice/regular" $R(=M^*)$ and then consider $M := R^*(= r^{-1})$.
- (∗) Second idea: If R is standard and "nice enough", then take R^{-1}.
- (∗) Third approach: Start with "nice enough" R and consider the so-called dual sequence D.

Dual - II

Let $\mathbf{a} = (a_p)_p \in \mathbb{R}_{>0}^{\mathbb{N}}$, then the *upper Matuszewska index* $\alpha(\mathbf{a})$ is defined by

$$\begin{aligned}\alpha(\mathbf{a}) &:= \inf\{\alpha \in \mathbb{R} : \frac{a_p}{p^\alpha} \text{ is almost decreasing}\} \\ &= \inf\{\alpha \in \mathbb{R} : \exists\, H \geq 1\ \forall\, 1 \leq p \leq q : \quad \frac{a_q}{q^\alpha} \leq H \frac{a_p}{p^\alpha}\},\end{aligned}$$

and the *lower Matuszewska index* $\beta(\mathbf{a})$ by

$$\begin{aligned}\beta(\mathbf{a}) &:= \sup\{\beta \in \mathbb{R} : \frac{a_p}{p^\beta} \text{ is almost increasing}\} \\ &= \sup\{\beta \in \mathbb{R} : \exists\, H \geq 1\ \forall\, 1 \leq p \leq q : \quad \frac{a_p}{p^\beta} \leq H \frac{a_q}{q^\beta}\}.\end{aligned}$$

Dual - III

These values give an alternative (more compact) possibility to formulate the main results concerning weighted entire classes:

(∗) Take $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\alpha(\mu) < 1$. Here $\mu = (\mu_p)_p$ with $\mu_p = M_p / M_{p-1}$.

(∗) If $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $\alpha(\mu) < +\infty$, then multiply M with an appropriate Gevrey-sequence.

(∗) Let, e.g., $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $\beta(\mu) > 1$ be given - then consider M^{-1}.

($\beta(\mu) > 1$ for $M \in \mathcal{LC}$ precisely means that M is strong non-quasianalytic.)

(∗) For G^s both indices coincide and are equal to s.

Dual - IV

(∗) Let $N \in \mathcal{LC}$ and consider the counting function

$$\Sigma_N(t) := |\{p \in \mathbb{N}_{>0} : \nu_p = N_p/N_{p-1} \leq t\}|, \quad t \geq 0.$$

(∗) The dual sequence D is defined by

$$\forall\, p \geq \nu_1 (\geq 1): \quad \delta_{p+1} := \Sigma_N(p), \quad \delta_{p+1} := 1 \quad -1 \leq p < \nu_1,$$

and so set $D_p := \prod_{i=0}^{p} \delta_i$.

(∗) By definition $D \in \mathcal{LC}$ with $1 = D_0 = D_1$.

(∗) The sequence $N_p = p!$ is self-dual.

Dual - V

The main preparatory result (J. Jiménez-Garrido, '18) in this context is:

Theorem

Let $N \in \mathcal{LC}$ be given. Assume that

$$\exists\, B \geq 1\ \forall\, p \in \mathbb{N}: \quad \nu_{p+1} \leq B\nu_p. \tag{3}$$

Then $\alpha(\nu) = \frac{1}{\beta(\delta)}$ *and* $\beta(\nu) = \frac{1}{\alpha(\delta)}$.

(3) is strictly weaker than $(M2)$/moderate growth and strictly stronger than $(M2)'$/derivation closedness.

Involving this information we are able to show...

Dual - VI

Theorem

Let $N \in \mathcal{LC}$ be given and let D be the dual sequence. We assume that:

($*$) $\beta(\nu) > 1$ *and*

($*$)

$$\exists\, B \geq 1\ \forall\, p \in \mathbb{N}: \quad \nu_{p+1} \leq B\nu_p.$$

Then there exists $L \in \mathbb{R}_{>0}^{\mathbb{N}}$ which is equivalent to D and such that L satisfies all requirements in order to apply the characterization for $\mathcal{E}_{[L]}$ in terms of the weighted entire space given by L^*.

If $1 < \beta(\nu) \leq \alpha(\nu) < +\infty$, then L satisfies (except normalization) the requirements from $(\mathfrak{F}_1)$, $(\mathfrak{F}_2)$, and $(\mathfrak{F}_4)$.

(∗) D. N. Nenning and G. Schindl, Ultradifferentiable classes of entire functions, Adv. in Op. Theory 8, art. no. 67, 2023; doi: 10.1007/s43036-023-00294-6.

(∗) M. V. Markin, On the strong smoothness of weak solutions of an abstract evolution equation III. Gevrey ultradifferentiability of order less than one, Applic. Analysis, 78 (1-2), 139–152, 2001; doi: 10.1080/00036810108840930.