Parametric ψdo with point-singularity in the covariable

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General aim

Describe a calculus of parameter-dependent ψdo that allows the construction of a parametrix for Toeplitz type ψdo which coincides with the inverse for large values of the parameter.

As a by-product we obtain a calculus that allows to recover the Grubb-Seeley resolvent trace expansion.

We deal with operators on $\mathbb{R}^n$ or closed Riemannian manifolds (of dimension ≥ 2).

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Important notation:

- Parameter $\mu \in \overline{\mathbb{R}}_+ = [0, +\infty)$,
- Covariable $\xi \in \mathbb{R}^n$ (to variable x).

For the most part of the talk we consider operators with constant coefficients. The results hold in general for operators with smooth (bounded) coefficients.

Resolvents of differential operators

Let $A = \sum_{|\alpha| \leq d} a_\alpha D_x^\alpha$. Then $\mu^d - A$ has symbol

$$a(\xi, \mu) = \mu^d - \sum_{|\alpha| \leq d} a_\alpha \xi^\alpha, \qquad a_d(\xi, \mu) = \mu^d - \sum_{|\alpha| = d} a_\alpha \xi^\alpha.$$

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Definition (Hörmander class): $a(\xi, \mu) \in S^d_{1,0}$, $d \in \mathbb{Z}$, if

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$a(\xi,\mu) \in S^d$ is polyhomogeneous/classical if

$$\boxed{a \sim \sum_\ell \chi(\xi,\mu) a_{d-\ell}, \qquad a_{d-\ell} \in S^{d-\ell}_{hom}.}$$

with $a_{d-\ell}$ positively homogeneous of degree $d-\ell$ in $(\xi,\mu) \neq 0$.

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Theorem: $a \in S^d$ elliptic $\Rightarrow$ Exists $b \in S^{-d}$ such that

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Observation: Above approach fails for ψdo: If

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There is a "breaking point" for the estimates, where one changes from $\langle\xi,\mu\rangle$ to $\langle\xi\rangle$.

Grubb's class and regularity number

Definition (Grubb 1980's): $a(\xi, \mu) \in S^{d,\nu}_{1,0}$, $d \in \mathbb{Z}$, $\nu \in \mathbb{R}$, if

$$|D^{\alpha}_{\xi} D^{j}_{\mu} a(\xi, \mu)| \lesssim \langle \xi, \mu \rangle^{d-|\alpha|-j} + \langle \xi \rangle^{\nu-|\alpha|} \langle \xi, \mu \rangle^{d-\nu-j}.$$

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$S^{d,\nu}$ subclass of polyhomogeneous symbols, i.e.,

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- $S^{d-\infty,\nu-\infty} = \cap_\ell S^{d-\ell,\nu-\ell} = S^{d-\nu}(\overline{\mathbb{R}}_+, \mathscr{S}(\mathbb{R}^n_\xi))$.
- $S^d = S^{d,+\infty} = \cap_{\nu\geq 0} S^{d,\nu}$ (infinite regularity)

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Remark: In general, homogeneous components are only defined for $\xi \neq 0$. If $\nu > 0$, $\sigma(a)$ extends by continuity to $(\xi, \mu) \neq 0$.

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Definition: $a \in S^{d,\nu}$, $\nu > 0$, is elliptic if $\sigma(a)$ never vanishes for $(\xi, \mu) \neq 0$.

Theorem: $a \in S^{d,\nu}$, $\nu > 0$, elliptic $\Rightarrow$ Exists $b \in S^{-d,\nu}$ such that

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Application: Resolvent of ψdo of positive integer order d.
In this case the regularity number is $\nu = d$.

Note: No concept of ellipticity for regularity number $\nu \leq 0$.

ψdo of Toeplitz type

Aim: We want to construct the inverse of ψdo of the form

$$P_1\,(\mu^d - a(D))\,P_0 : \; P_0\big(H^s(\mathbb{R}^n,\mathbb{C}^M)\big) \longrightarrow P_1\big(H^{s-d}(\mathbb{R}^n,\mathbb{C}^M)\big)$$

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Problem: Here, the regularity number is $\nu = 0$.
Cannot use Grubb's class $S^{d,0}$.

Some background

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The BdM calculus is made by operators of the form

$$\begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} H^s(M, E_0) \\ \oplus \\ H^s(\partial M, F_0) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}(M, E_1) \\ \oplus \\ H^{s-\mu}(\partial M, F_1) \end{matrix} .$$

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Observation: Not every elliptic differential operator A can be completed to an elliptic bvp $\begin{pmatrix} A \\ T \end{pmatrix}$ in BdM's calculus.

Some background

Example: $D \subset \mathbb{R}^2$ unit-disc and

$$\begin{pmatrix} \overline{\partial} \\ P \circ \cdot|_{\partial D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \overline{\partial} \\ \cdot|_{\partial D} \end{pmatrix} : \; H^s(D) \longrightarrow \begin{array}{c} H^{s-1}(D) \\ \bigoplus \\ P\big(H^{s-1/2}(\partial D)\big) \end{array}$$

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Schulze, Shatalov, Sternin ($\sim$ 2000): Systematic study of ellipticity and Fredholm property of operators

$$\boxed{\begin{pmatrix} A \\ P_1 T \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} A & K \\ T & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P_0 \end{pmatrix},}$$

with zero-order projections P_j on the boundary.
This results in an "extended" BdM calculus.

Some background

Example: The Stokes operator on a bounded domain $\Omega \subset \mathbb{R}^n$ is

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \Delta \\ \cdot|_{\partial\Omega} \end{pmatrix} P : \; H^s_\sigma(\Omega, \mathbb{C}^n) \longrightarrow \begin{matrix} H^{s-2}_\sigma(\Omega, \mathbb{C}^n) \\ \oplus \\ H^{s-1/2}_\nu(\partial\Omega, \mathbb{C}^n) \end{matrix}$$

with the solenoidal vector-fields on Ω and "tangential vector-fields" on the boundary.

P is the Helmholtz projection, Q "projection along normal".

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Question: Given some kind of "calculus of ψdo", what can we say about Toeplitz type operators

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Theorem: If the calculus is "nice" i.e.,

- ellipticity, existence of a parametrix, and Fredholm property are equivant to the invertibility of principal symbol(s),
- the calculus is closed under taking the formal adjoint,

then all descents to the Toeplitz calculus, using principal symbol(s)

$$\boxed{\sigma(A') = \sigma(P_1)\,\sigma(A)\,\sigma(P_0) : \ \mathrm{range}\,\sigma(P_0) \longrightarrow \mathrm{range}\,\sigma(P_1).}$$

A "geometric" description of $S^{d,\nu}$

Homogeneous components:

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Question: To which space corresponds $S^{d-\ell,\nu-\ell}_{hom}$?

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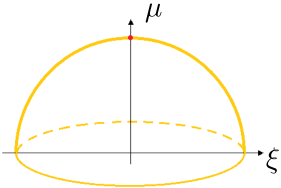

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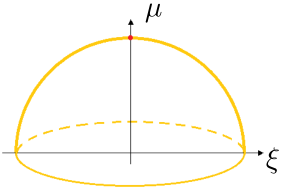

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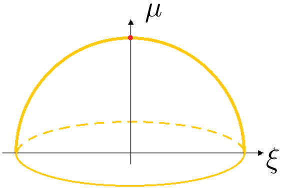

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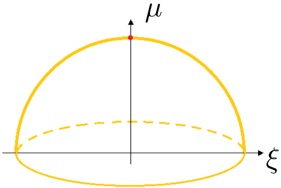

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Definition: $\widetilde{S}^{d,\nu}_{hom} \cong r^\nu C_B^\infty(\widehat{\mathbb{S}}^n_+)$. $\rightsquigarrow$ Yields full class $\widetilde{S}^{d,\nu}$.

Theorem: $S^{d,\nu} = S^d + \widetilde{S}^{d,\nu}$.

In particular: $S^{d,\nu} = \widetilde{S}^{d,\nu}$ whenever $\nu \leq 0$.

Symbols with Taylor asymptotics

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Lemma: Let $\widehat{a} \in r^\nu C^\infty_{\mathbf{T}}(\widehat{\mathbb{S}}^n_+)$. Then $1/\widehat{a} \in r^{-\nu} C^\infty_{\mathbf{T}}(\widehat{\mathbb{S}}^n_+)$ iff

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- principal symbol $\sigma(a) \in \widetilde{\mathbf{S}}^{d,\nu}_{hom}$,
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Lemma: Let $a \in \widetilde{S}_T^{d,\nu}$ have non-vanishing principal symbols.

Then there exists a $b \in \widetilde{S}_T^{-d,-\nu}$ such that

$$a(\mu, D)b(\mu, D) = 1 + r(\mu, D), \quad r \in \widetilde{S}^{0-\infty,0-\infty} = S^0(\overline{\mathbb{R}}_+, S^{-\infty}(\mathbb{R}^n_\xi)).$$

Symbols with Taylor asymptotics

We could implement a symbol class $\widetilde{S}_T^{d,\nu} \subset \widetilde{S}^{d,\nu}$ with

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Observation: We still need a condition at $\mu = +\infty$ such that we get a remainder decaying in μ !

The limit-operator

Key observation: Let $a \in \widetilde{\mathbf{S}}^{d,\nu}_{hom}$. Then, for every N,

$$a(\xi,\mu) = \sum_{j=0}^{N-1} \underbrace{|\xi|^{\nu+j}\widehat{a}_j\Big(\frac{\xi}{|\xi|}\Big)}_{=:a^\infty_{\nu+j}(\xi)\,\in\, S^{\nu+j}_{hom}(\mathbb{R}^n\setminus 0)} \underbrace{|\xi,\mu|^{d-\nu-j}}_{\in S^{d-\nu-j}_{hom}} \mod \widetilde{S}^{d,\nu+N}_{hom}.$$

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All of this can be incorporated in a resulting symbol class:

Definition: $a \in \widetilde{\mathbf{S}}^{d,\nu}$ if

- has homogeneous components in $\widetilde{\mathbf{S}}^{d-\ell,\nu-\ell}_{hom}$,
- admits an expansion $a \sim \sum_j a^\infty_{\nu+j}(\xi)\,[\xi,\mu]^{d-\nu-j}$ with symbols $a^\infty_{\nu+j} \in S^{\nu+j}(\mathbb{R}^n)$.

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Definition: $a^\infty_\nu(D)$ is the principal limit operator of $a \in \widetilde{\mathbf{S}}^{d,\nu}$.

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Remark: The homogeneous principal symbol of the principal limit-operator coincides with the principal angular symbol.

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Definition: $a \in \widetilde{\mathbf{S}}^{d,\nu}$ is called elliptic if its homogeneous principal symbol never vanishes and its principle limit-operator is invertible.

Theorem: If $a \in \widetilde{\mathbf{S}}^{d,\nu}$ is elliptic then there exists a $b \in \widetilde{\mathbf{S}}^{-d,-\nu}$ such that

$$\boxed{a(\mu, D)b(\mu, D) = b(\mu, D)a(\mu, D) = 1, \qquad \mu >> 1.}$$

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This result is "inherited" by ψdo of Toeplitz type.

Inverses of Toeplitz type ψdo

Consider an operators of the form

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- $P_1\, a_0^\infty(D)\, P_0:\ P_0(H^s(\mathbb{R}^n,\mathbb{C}^M)) \to P_1(H^s(\mathbb{R}^n,\mathbb{C}^M))$

Then there exists $B(\mu) = P_0\, b(D,\mu)\, P_1$ with $b \in \widetilde{\mathbf{S}}^{-d,0}$ s.t.

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Remark: In case $a(D,\mu) = \mu^d - a(D)$, $a(\xi) \in S^d$, the second condition is equivalent to the invertibility of

$$P_1:\ P_0(H^s(\mathbb{R}^n,\mathbb{C}^M)) \to P_1(H^s(\mathbb{R}^n,\mathbb{C}^M)).$$

Resolvent trace asymptotics

Let $P \in S^d(\mathbb{R}^n)$, $d \in \mathbb{N}$, be elliptic w.r.t. a closed sector $\Lambda \subset \mathbb{C}$.
We are interested in

$$\boxed{\mathrm{Tr}\big(Q(\lambda - P)^{-N}\big), \quad |\lambda| \to \infty,}$$

where $Q \in S^\omega(\mathbb{R}^n)$, $\omega \in \mathbb{R}$, and $\omega - Nd < -n$.

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Theorem (Grubb-Seeley '95): If $k(x, y, \lambda)$ is the distributional kernel of the above operator, there exist $c_j, c_j', c_j'' \in C_b^\infty(\mathbb{R}^n)$ such that

$$k(x, x, \lambda) \sim \sum_{j=0}^{+\infty} c_j(x)\lambda^{\frac{n+\omega-j}{d} - N} + \sum_{j=0}^{+\infty} \big(c_j'(x)\log\lambda + c_j''(x)\big)\lambda^{-N-j},$$

uniformly for $\lambda \in \Lambda$ with $|\lambda| \longrightarrow +\infty$.

Resolvent trace asymptotics

Lemma: $a(\xi,\mu) \in \widetilde{\mathbf{S}}^{d,\nu}$, $d-\nu \leq 0$, has a Grubb-Seeley expansion:

$$a(\xi,\mu) \sim_{\mu\to+\infty} \sum_\ell q_\ell(\xi)\,\mu^{d-\nu-\ell}, \qquad q_\ell \in S^{\nu+\ell}(\mathbb{R}^n).$$

If also $d < -n$ this results in an expansion

$$k_a(x,x,\mu) \sim \sum_{j=0}^{+\infty} c_j(x)\mu^{d-j+n} + \sum_{\ell=0}^{+\infty} \big(c'_\ell(x)\log\mu + c''_j(x)\big)\mu^{d-\nu-\ell}.$$

Resolvent trace asymptotics

Proof of the Lemma:

Start out from expansion the

$$a(\xi,\mu) \sim \sum_j a^{\infty}_{\nu+j}(\xi)\,[\xi,\mu]^{d-\nu-j}, \qquad a^{\infty}_{\nu+j}(\xi) \in S^{\nu+j},$$

and insert expansions

$$[\xi,\mu]^m \sim \sum_\ell \zeta_{m,j}(\xi)\,\mu^{m-\ell}, \qquad m \le 0,$$

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Repeat the proof of Grubb-Seeley to get expansion of $k_a(x,x,\mu)$.

Resolvent trace asymptotics

Proof of the resolvent kernel expansion:

Using the $\widetilde{\mathbf{S}}$-calculus in the, one can show that

$$\boxed{Q(\mu^d - e^{i\theta} P)^{-N} \in \widetilde{\mathbf{S}}^{\omega - Nd, \omega}}$$

uniformly in θ with $e^{i\theta} \in \Lambda$.

Combine this with the previous lemma.

The manifold case

The calculus can be defined for any closed Riemannian manifold M (and operators acting on sections of smooth vector bundles).

- Coordinate-invariance of $\widetilde{\mathbf{S}}^{d,\nu}$.
- Standard patching of local operators with partion of unity.

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- Coordinate-invariance of $\widetilde{\mathbf{S}}^{d,\nu}$.
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Theorem: The above local expansion has a global analogue:

$$A(\mu) \sim \sum_{j=0}^{\infty} A^{\infty}_{\nu+j}\, \Lambda^{d-\nu-j}(\mu), \qquad A^{\infty}_{\nu+j} \in L^{\nu+j}(M),$$

where the $\Lambda^{\alpha}(\mu)$ are ψdo of order α with principal symbol

$$\sigma(\Lambda^{\alpha})(\xi,\mu) = (|\xi|^2 + \mu^2)^{\alpha/2}.$$

The manifold case

The ellipticity on M involves:

- homogeneous principal symbol,
- limit operator: $A_\nu^\infty(D) \in L^\nu(M)$.

As a subordinate principal symbol we have

- principal angular symbol $\sigma(A_\nu^\infty)$.

Some papers

- J. S., *Parametric pseudodifferential operators with point-singularity in the covariable*. Annals of Global Analysis and Geometry **61** (2022), 553–592.
- J. S., *Singular Green operators in the edge algebra formalism*. Journal of Mathematical Analysis and Applications **511** (2022), Paper No. 126041, 39 pp.
- B.-W. Schulze, J. S., *Elliptic complexes on manifolds with boundary*, Journal of Geometric Analysis **29** (2019), no. 1, 656-706.
- J. S., *Parameter-dependent pseudodifferential operators of Toeplitz type*. Annali di Matematica Pura ed Applicata **194** (2015), no. 1, 145-165.
- J. S., *Ellipticity in pseudodifferential algebras of Toeplitz type*. Journal of Functional Analysis **263** (2012), no. 5, 1408-1434.

Thank you for your attention !