Solving non-Markovian Stochastic Control Problems driven by Wiener Functionals

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Issues to be addressed

Let $\xi : C([0,T];\mathbb{R}^n) \to \mathbb{R}$ be a Borel functional, let $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be a reference filtration generated by a multi-dimensional Brownian motion. Let $U_t^T; 0 \le t < T$ be the set of $\mathbb{F}$-predictable controls defined over $(t,T]$ and taking values on a compact subset $\mathbb{A}$.

Let $\{X^u; u \in U_0^T\}$ be a family of $\mathbb{F}$-adapted controlled processes.

We are interested in the stochastic optimal control problem

$$\sup_{\phi \in U_0^T} \mathbb{E}\Big[\xi(X^\phi)\Big]$$

in the following sense:

Issues to be addressed

QUESTION: For a given error bound $\epsilon > 0$, how to design a **numerical scheme to compute** ϵ-optimal controls $\phi^{*,\epsilon}$, i.e.,

$$\mathbb{E}\left[\xi(X^{\phi^{*,\epsilon}})\right] \geq \sup_{\phi \in U_0^T} \mathbb{E}\left[\xi(X^{\phi})\right] - \epsilon.$$

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This is an old, classical and (at some extent) well-understood question in case X^{ϕ} is a controlled Markov process. Answer:

- PDE techniques (Hamilton-Jacobi-Bellman) and Monte Carlo schemes.
- Markov chain approximations.

Beyond Markovian case: Path-dependent SDEs driven by Brownian motion

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{path-dependent functional}} \, dt + \overbrace{\sigma(t, X^u_t, u(t))}^{\text{path-dependent functional}} \, dB(t),$$

where B is a Brownian motion and $X^u_t := \{X^u(s); 0 \leq s \leq t\}$.

- Characterizations of the value process.
  - 2BSDEs: Nutz (2012)
  - Randomization approach: Fuhrman and Pham (2015)
  - 2BSDE and path-dependent PDEs: Possamaï, Tan and Zhou (2018).
  - Functional HJB-type equation: Qiu, J. (2018).

Beyond Markovian case: Path-dependent SDEs driven by Brownian motion

- Numerical methods for path-dependent SDEs driven by Brownian motion.
  - G-expectations: Dolinsky (2012).
  - Monte Carlo scheme: Tan (2014).
  - Monotone scheme for path-dependent PDE: Zhang and Zuo (2014), Ren and Tan (2016).
  - Policy iteration algorithm: Possamaï and Tangpi (2024).

Controlled systems with non-trivial memory

In the fully non-Markovian case, feasible numerical approximation schemes are very challenging !

Typical non-trivial example:

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{non-anticipative functional}} dt + \overbrace{\sigma(t, X^u_t, u(t))dB^H(t)}^{\text{fully non-Markovian}}$$

where B^H is a fractional Brownian motion with exponent $H \in (0,1)$ given by

$$B \mapsto B^H(\cdot) = \int_0^{\cdot} K_H(\cdot, u)dB(u)$$

Controlled systems with non-trivial memory

$$B \mapsto B^H(\cdot) = \int_0^{\cdot} K_H(\cdot, u) dB(u)$$

- Highly singular infinite-dimensional map for $0 < H < \frac{1}{2}$
- Regular infinite-dimensional map for $\frac{1}{2} < H < 1$
- If $H \neq \frac{1}{2}$, one cannot reduce it to a Markovian situation without adding infinitely many degrees of freedom.

Related literature

Optimality characterization:

- Maximum principle
  - Biagini, Hu, Oksendal and Sulem (2002)
  - Han, Hu and Song (2013).
- Lifting approach, relaxed controls:
  - Path-dependent-type PDE: Viens and Zhang (2018)
  - Control in UMD spaces: Di Nunno and Giordano (2023), Chakraborty, Honnappa and Tindel (2024).
  - Relaxed controls: Cárdenas, Pulido and Serrano (2025).

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  - Relaxed controls: Cárdenas, Pulido and Serrano (2025).
- An attempt to a numerical scheme:
  - Infinite-dimensional Ricatti equations for linear-quadratic problems: Jaber, Miller and Pham (2021).

Typical examples we have in mind

In this talk, we will present a **concrete numerical scheme for computing near optimal controls** for controlled processes adapted to the Brownian filtration beyond the linear-quadratic cases:

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{path-dependent functional}} \; dt + \overbrace{\sigma(t, X^u_t, u(t))}^{\text{path-dependent functional}} \; dB(t), \qquad (1)$$

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{non-linear functional}} \; dt + \overbrace{\sigma dB^H(t)}^{\text{fully non-Markovian}} \qquad (2)$$

and

$$\begin{cases} dX^u(t) = X^u(t)\mu\big(u(t)\big)dt + X^u(t)\vartheta\big(Z(t), u(t)\big)dB_t \\ \underbrace{dZ(t) = \Phi(dt, dZ(t), dB^H(t))}_{\text{SDE driven by } B^H}, \end{cases} \qquad (3)$$

where B^H is the fractional Brownian motion with $H \in (0, \frac{1}{2})$.

The idea of the method

We construct a suitable underlying **imbedded** discrete structure (we *do not* lift to an infinite-dimensional Markovian system) inherited from the Brownian motion which allows us to construct a **discrete-time backward dynamic programming equation** associated with

$$V(t,u) = \operatorname*{ess\,sup}_{\phi;\phi=u \text{ on } [0,t]} \mathbb{E}\big[\xi\big(X^{\phi}\big)|\mathcal{F}_t\big];\ 0\le t\le T. \tag{4}$$

where $V(T,u)=\xi(X^u)$ and $V(0)=\sup_{\phi\in U_0^T}\mathbb{E}[\xi(X^{\phi})]$.

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where $V(T,u) = \xi(X^u)$ and $V(0) = \sup_{\phi\in U_0^T} \mathbb{E}[\xi(X^\phi)]$.

- The solution of our discrete-time dynamic programming equation provides a near optimal stochastic control for the original problem (4).
- Optimal controls resulting from our dynamic programming equation can be numerically computed by Machine/Deep Learning techniques.

Contributions

Our contribution relies on:

1. Development of a numerical scheme for computing near-optimal controls for (possibly) fully non-Markovian controlled processes.
2. Explicit rates of convergence are provided under rather weak conditions.
3. Closed/open-loop optimal controls are obtained and classified according to the strength of the possibly underlying non-Markovian memory.

Main References

- Leão, D., O-A. and Souza, F. (2024). Solving non-Markovian Stochastic Control Problems driven by Wiener Functionals. *AAP*.
- Leão,D. O-A. and Simas, A. B. (2018). A weak version of path-dependent functional Itô calculus. *AOP*.
- Leão,D. O-A. and Simas, A. B. (2018). Weak differentiability of Wiener functionals and occupation times. *BSM*.
- O-A and Souza, F.A. (2020) L^p uniform random walk-type approximation for fractional Brownian motion with Hurst exponent $0 < H < \frac{1}{2}$. *EJP*.

Basic structure of $\mathscr{D}$

We are going to fix a d-dimensional Brownian motion $B = \{B^1, \ldots, B^d\}$ on $(\Omega, \mathbb{F}, \mathbb{P})$, where Ω is the space $C(\mathbb{R}_+; \mathbb{R}^d) := \{f : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous}\}$, $\mathbb{P}$ is the Wiener measure on Ω such that $\mathbb{P}\{B(0) = 0\} = 1$ and $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is the usual $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion.

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In the sequel,

$$u \mapsto X^u$$

is a controlled $\mathbb{F}$-adapted continuous process defined on U_0^T.

Standing assumptions

In the sequel, we denote $\mathbf{D}_{n,T} := \{h : [0, T] \to \mathbb{R}^n \text{ with càdlàg paths}\}$. We now present the two standing assumptions of this talk.

Assumption (A1): The payoff $\xi : \mathbf{D}_{n,T} \to \mathbb{R}$ satisfies the following regularity assumption: There exists $\gamma \in (0, 1]$ and a constant $\|\xi\| > 0$ such that

$$|\xi(f) - \xi(g)| \leq \|\xi\| \|f - g\|_\infty^\gamma,$$

for every $f, g \in \mathbf{D}_{n,T}$, where $\|f\|_\infty := \sup_{0 \leq t \leq T} |f(t)|$.

Assumption (B1): There exists a constant C such that

$$\mathbb{E}\|X^u - X^\eta\|_\infty^2 \leq C\mathbb{E} \int_0^T |u(s) - \eta(s)|^2 ds, \tag{5}$$

for every $u, \eta \in U_0^T$.

We start by constructing a sequence $\mathcal{T} := \{T_n^k; n \geq 0\}$ of hitting times which will be the basis for our discretization scheme. Fix a sequence $\epsilon_k \downarrow 0$ as $k \to +\infty$. We set $T_0^k := 0$ and

$$T_n^k := \inf\{T_{n-1}^k < t < \infty; |B(t) - B(T_{n-1}^k)| = \epsilon_k\}, \quad n \geq 1.$$

Then, we define $A^k := (A^{k,1}, \cdots, A^{k,d})$ by

$$A^{k,j}(t) := \sum_{n=1}^{\infty} \Big(B^j(T_n^k) - B^j(T_{n-1}^k)\Big) \mathbb{1}_{\{T_n^k \leq t\}}; \;\; t \geq 0, \; j = 1, \ldots, d,$$

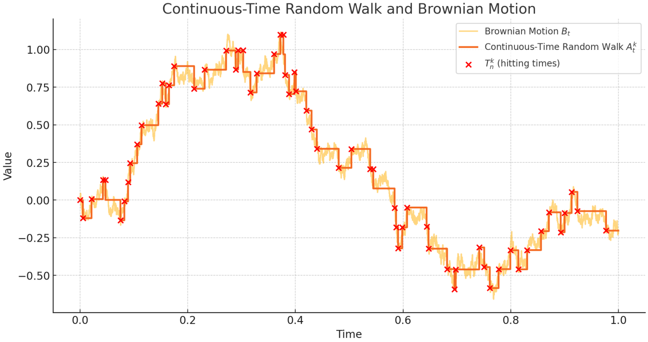
for integers $k \geq 1$.

The underlying discrete skeleton

In the one dimensional case, we have

$$T_n^k = \inf\left\{T_{n-1}^k < t < \infty; |B(t) - B(T_{n-1}^k)| = \varepsilon_k\right\}, \quad n \geq 1.$$

$\{A^k(T_n^k) - A^k(T_{n-1}^k); n \geq 1\}$ is an iid sequence of Bernoulli variables.



The underlying discrete skeleton

Let $\mathbb{F}^k = (\mathcal{F}_t^k)_{t\geq 0}$ be the filtration generated by A^k. One can check

$$\mathcal{F}^k_{T^k_n} = \sigma\Big(\Delta T^k_i, \Delta A^k(T^k_i); 1 \leq i \leq n\Big),$$

where $\Delta T^k_n := \underbrace{T^k_n - T^k_{n-1}}_{\text{Burq Jones algorithm (2008)}} \stackrel{d}{=} T^k_1$ and

$$\Delta A^k(T^k_n) := \underbrace{A^k(T^k_n) - A^k(T^k_{n-1})}_{\text{Bernoulli (1-dim) or conditioned truncated Gaussian (d-dim)}}.$$

Definition

The structure $\mathscr{D} = \{\mathcal{T}, A^k; k \geq 1\}$ is called a **discrete-type skeleton** for the Brownian motion.

The number of steps

Let us define

$$e(k,T) := \left\lceil \frac{\epsilon_k^{-2} T}{\chi_d} \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer greater or equal to $x \geq 0$ and

$$\chi_d := \mathbb{E} \min\{\tau^1, \ldots, \tau^d\},$$

where $(\tau^j)_{j=1}^d$ is an iid sequence of random variables with distribution $\inf\{t > 0; |W(t)| = 1\}$ for a real-valued standard Brownian motion W.

Discretizing the set of controls

Let $U_0^{k,e(k,T)}$ be the set of $\mathbb{F}^k$-predictable processes of the form

$$v^k(t) = \sum_{j=1}^{e(k,T)} v_{j-1}^k \mathbb{1}_{\{T_{j-1}^k < t \leq T_j^k\}},$$

where for each $j = 1, \ldots, e(k, T)$, v_{j-1}^k is an $\mathbb{A}$-valued $\mathcal{F}_{T_{j-1}^k}^k$-measurable random variable.

Controlled imbedded discrete structures

The structure $\mathscr{D}$ is dense in the Wiener space in the following sense:

Theorem Leão, O-A (2018, 2024)

For a given controlled process $u \mapsto X^u$ satisfying Assumption B1, we can associate a discrete type structure $\mathcal{X} = \big((X^k)_{k\geq 1}, \mathscr{D}\big)$ of the following form: For each $\phi \in U_0^{k,e(k,T)}$,

$$X^{k,\phi}(t) = \sum_{n=0}^{\infty} X^{k,\phi}(T_n^k)\mathbb{1}_{\{T_n^k \leq t \wedge T^k_{e(k,T)} < T^k_{n+1}\}},$$

where $X^{k,\phi}(T_n^k)$ is $\mathcal{F}^k_{T_n^k}$-measurable for every $n \geq 0$ and $k \geq 1$. Moreover, there exists a positive sequence $h_k \downarrow 0$ such that

$$\sup_{\phi \in U_0^{k,e(k,T)}} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_\infty \lesssim h_k,$$

for $k \geq 1$. A pair $(X, \mathcal{X})$ is called an **imbedded discrete structure** for X.

Typical examples of controlled imbedded discrete structures

(Controlled path-dependent SDEs)

$$\begin{aligned} X^{k,v^k}(T^k_n) &= X^{k,v^k}(T^k_{n-1}) + \alpha\Big(T^k_{n-1}, X^{k,v^k}_{T^k_{n-1}}, v^k_{n-1}\Big)\Delta T^k_n \\ &+ \sigma\Big(T^k_{n-1}, X^{k,v^k}_{T^k_{n-1}}, v^k_{n-1}\Big)\Delta A^k(T^k_n), \end{aligned}$$

for $n \geq 1$.

Typical examples of controlled imbedded discrete structures

(Controlled path-dependent SDEs)

$$X^{k,v^k}(T_n^k) = X^{k,v^k}(T_{n-1}^k) + \alpha\Big(T_{n-1}^k, X^{k,v^k}_{T_{n-1}^k}, v_{n-1}^k\Big)\Delta T_n^k$$
$$+ \sigma\Big(T_{n-1}^k, X^{k,v^k}_{T_{n-1}^k}, v_{n-1}^k\Big)\Delta A^k(T_n^k),$$

for $n \geq 1$.

(Controlled path-dependent SDEs driven by FBM)

$$X^{k,v^k}(T_n^k) = X^{k,v^k}(T_{n-1}^k) + \alpha\Big(T_{n-1}^k, X^{k,v^k}_{T_{n-1}^k}, v_{n-1}^k\Big)\Delta T_n^k$$
$$+ \sigma\Delta B_H^k(T_n^k),$$

B_H^k is a $\mathscr{D}$-discretization of B^H.

The FBM imbedded discrete structure

Let $K_H(t,s) = K_{H,1}(t,s) + K_{H,1}(t,s)$ be the classical Volterra-kernel of FBM,

$$K_{H,1}(t,s) := c_{H,1} t^{H-\frac{1}{2}} s^{\frac{1}{2}-H} (t-s)^{H-\frac{1}{2}},$$

$$K_{H,2}(t,s) := c_{H,2} s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du$$

for $s < t$ and constants $c_{H,1}$ and $c_{H,2}$.

Let C_0^λ be the space of Hölder continuous functions f such that $f(0) = 0$. For each $f \in C_0^\lambda$, we set

$$\begin{aligned}(\Lambda_H f)(t) &:= \int_0^t \partial_s K_{H,1}(t,s)[f(t) - f(s)]ds \\ &- \int_0^t \partial_s K_{H,2}(t,s) f(s) ds.\end{aligned}$$

The FBM imbedded discrete structure

Theorem O-A, Souza (2020)

Any FBM with exponent $0 < H < \frac{1}{2}$ on a time interval $[0, T]$ can be represented by $\Lambda_H B$ for a real-valued standard Brownian motion B.

The FBM imbedded discrete structure

Let $\bar{t}_k := \max\{T^k_n; T^k_n \leq t\}$ and $\bar{t}^+_k := \min\{T^k_n; \bar{t}_k < T^k_n\} \wedge T$.

$$\begin{aligned} B^k_H(t) &:= \int_0^{\bar{t}_k} \partial_s K_{H,1}(\bar{t}_k, s)[A^k(\bar{t}_k) - A^k(\bar{s}^+_k)]ds \\ &- \int_0^{\bar{t}_k} \partial_s K_{H,2}(\bar{t}_k, s)A^k(s)ds. \end{aligned}$$

Theorem O-A, Souza (2020)

Fix $0 < H < \frac{1}{2}$ and $p \geq 1$. For every pair (δ, λ) such that $\max\{0, 1 - \frac{pH}{2}\} < \delta < 1$, $\lambda \in \left(\frac{1-H}{2}, \frac{1}{2} + \frac{\delta-1}{2}\right)$, we have

$$\mathbb{E}\|B^k_H - B_H\|^p_\infty \lesssim_{p,\delta,\lambda,H,T} \epsilon_k^{p(1-2\lambda)+2(\delta-1)} \to 0$$

as $k \to +\infty$.

The approximated value process

Notation: $\xi_{X^k}(u) := \xi(X^{k,u})$ for a given controlled imbedded discrete structure $u \mapsto X^{k,u}$.
We set

$$V^k(T_n^k, u) := \operatorname*{ess\,sup}_{\phi \in U_n^{k,e(k,T)}} \mathbb{E}\Big[\xi_{X^k} \overbrace{(u \otimes_n \phi)}^{\text{concatenation}} \Big| \mathcal{F}^k_{T_n^k}\Big], \tag{6}$$

for $n = 1, \ldots, e(k,T) - 1$, with boundary conditions

$$V^k(0) := V^k(0,u) := \sup_{\phi \in U_0^{k,e(k,T)}} \mathbb{E}\big[\xi_{X^k}(\phi)\big]$$

and

$$V^k(T^k_{e(k,T)}, u) := \xi_{X^k}(u).$$

Next, we will construct a pathwise computable version of (6).

The approximated value process

Proposition

For each $u^k \in U_0^{k,e(k,T)}$, the discrete-time value process $V^k(\cdot, u^k)$ satisfies

$$V^k(T^k_{e(k,T)}, u^k) = \xi_{X^k}(u^k) \ a.s$$

$$V^k(T^k_n, u^k) = \overbrace{\operatorname{ess\,sup}}^{\text{sup}}_{\theta^k_n \in U_n^{k,n+1}} \mathbb{E}\left[V^k\left(T^k_{n+1}, u^{k,n-1} \otimes_n \theta^k_n\right) \mid \mathcal{F}^k_{T^k_n} \right], \quad (7)$$

for $0 \le n \le e(k,T) - 1$.

The approximated value process

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for $0 \leq n \leq e(k,T) - 1$.

We can actually prove that we can replace **esssup** by the **sup** in (7) by using analytic set theory techniques and the closed form expression for the law of $(\Delta T^k_1, \Delta A^k(T^k_1))$.

Intuition of aggregation

For a given control $u^k \in U_0^{k,e(k,T)}$ and a controlled structure $\mathcal{X} = ((X^k)_{k\geq 1}, \mathscr{D})$, we set

$$\mathcal{Y}_j^{k,u^k} := \Big(\mathcal{A}_1^k, \Delta X^{k,u^k}(T_1^k)), \ldots, \mathcal{A}_j^k, \Delta X^{k,u^k}(T_j^k))\Big),$$

for $1 \leq j \leq e(k,T)$. Here $\mathcal{A}_j^k := (\Delta T_j^k, \Delta A^k(T_j^k))$. The value functionals can be represented by a big functional

$$V^k(T_n^k, u^k) = \sup_{\theta\in\mathbb{A}} \int_{\mathbb{W}^k} \Phi\Big(\mathcal{Y}_{n-1}^{k,u^k}, \mathcal{X}_n^k(\theta, \mathcal{Y}_{n-1}^{k,u^k}, w^k)\Big)\nu^k(dw^k)$$

where $\mathfrak{X}_n^k$ is the jump of the $\mathscr{D}$-controlled state as step n and ν^k is the law of $(\Delta T_1^k, \Delta A^k(T_1^k))$ taking values on a set $\mathbb{W}^k$.

The Dynamic Programming Principle

Theorem Leão, O-A (2024)-Pathwise Dynamic Programming Equation

Let ν^k be the law of $(\Delta T_1^k, \Delta A^k(T_1^k))$. Starting from a given controlled state (standard terminal condition)

$$\mathbb{V}^k_{e(k,T)}(\mathbf{o}^k_{e(k,T)}) = \xi\big(\gamma^k_{e(k,T)}(\mathbf{o}^k_{e(k,T)})\big),$$

the value functionals ($\mathscr{D}$-version of the original value process) satisfy

$$V^k(T_j^k, u) = \mathbb{V}^k_j(\mathcal{Y}^{k,u}_j)$$

where

$$\begin{aligned} \mathbf{U}^k_j \qquad \overbrace{(\mathbf{o}^k_j, \theta)}^{\text{training data,control}} \qquad &:= \int_{\mathbb{W}_k} \mathbb{V}^k_{j+1}\Big(\mathbf{o}^k_j, \mathfrak{X}^k_{j+1}(\theta, \mathbf{o}^k_j, w^k)\Big)\nu^k(dw^k) \\ \mathbb{V}^k_j(\mathbf{o}^k_j) \quad &:= \sup_{\theta\in\mathbb{A}} \mathbf{U}^k_j(\mathbf{o}^k_j, \theta),\ j = e(k,T)-1,\dots,0, \end{aligned}$$

where $\mathfrak{X}^k_{j+1}$ is the jump of the $\mathscr{D}$-controlled state as step $j+1$.

The Dynamic Programming Principle

For a given $\epsilon > 0$, compute $C^{\epsilon}_{k,j} : \mathbb{H}^j_k \to \mathbb{A}$ (**via deep/reinforcement learning techniques**) such that

$$\mathbb{V}^k_j(\mathbf{o}^k_j) \leq \int_{\mathbb{W}_k} \mathbb{V}^k_{j+1}\Big(\mathbf{o}^k_j, \mathfrak{X}^k_{j+1}(C^{\epsilon}_{k,j}(\mathbf{o}^k_j), \mathbf{o}^k_j, w^k)\Big)\nu^k(dw^k) + \epsilon, \quad (8)$$

for every $\mathbf{o}^k_j$ training data, where $j = e(k,T) - 1, \ldots, 1$. Let $\eta_k(\epsilon) = \frac{\epsilon}{e(k,T)}$ and $u^k \in U^{k,e(k,T)}_0$. Define $\phi^{k,\eta_k(\epsilon)}_j$ as follows

$$\phi^{k,\eta_k(\epsilon)}_j = C^{\eta_k(\epsilon)}_{k,j}(\mathcal{Y}^{k,u^k}_j); j = e(k,T) - 1, \ldots, 0.$$

The control

$$\phi^{*,k,\epsilon} := (\phi^{k,\eta_k(\epsilon)}_0, \phi^{k,\eta_k(\epsilon)}_1, \ldots, \phi^{k,\eta_k(\epsilon)}_{m-1})$$

realizes

$$\sup_{\phi \in U^{k,e(k,T)}_0} \mathbb{E}\big[\xi_{X^k}(\phi)\big] \leq \mathbb{E}\big[\xi_{X^k}(\phi^{*,k,\epsilon})\big] + \epsilon.$$

Rate of convergence of the numerical scheme

Theorem Leão, O-A (2024)

Let us consider a pair $(X, \mathcal{X})$, $\mathcal{X} = ((X^k)_{k\geq 1}, \mathscr{D})$ such that there exists a positive sequence $h_k \downarrow 0$ such that

$$\sup_{\phi \in U_0^{k,e(k,T)}} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_{\infty} \lesssim h_k, \tag{9}$$

for $k \geq 1$. Let $V^k(0) := \sup_{u^k \in U_0^{k,e(k,T)}} \mathbb{E}[\xi_{X^k}(u^k)]; k \geq 1$.
Then, for a given $\epsilon > 0$ and $\beta \in (0,1)$, there exists a constant C which depends on $\beta, \|\xi\|_{\gamma}$ and Assumption (B1) such that

$$\Big|\sup_{\phi \in U_0^T} \mathbb{E}[\xi_X(\phi)] - V^k(0)\Big| \leq C \underbrace{\{h_k^{\gamma} + \epsilon_k^{\gamma\beta}\}}_{\text{Euler+Large deviations}} + \epsilon, \tag{10}$$

for every $k \geq 1$.

Rate of convergence of the numerical scheme

Continuation of Theorem Leão, O-A (2024)

For a given $\epsilon > 0$ and $k \geq 1$, let $\phi^{*,k,\epsilon} \in U_0^{k,e(k,T)}$ be a near optimal control associated with the **discrete-time control problem** computed before which realizes

$$\mathbb{E}[\xi_{X^k}(\phi^{*,k,\epsilon})] > V_k(0) - \frac{\epsilon}{3};\ \ k \geq 1.$$

Then, $\phi^{*,k,\epsilon} \in U_0^T$ is a near optimal control for the Brownian motion driving stochastic control problem, i.e.,

$$\mathbb{E}[\xi_X(\phi^{*,k,\epsilon})] > \sup_{\phi \in U_0^T} \mathbb{E}[\xi_X(\phi)] - \epsilon, \tag{11}$$

for every k sufficiently large.

Concrete cases

Proposition Leão, O-A (2024)

Let X^u be the controlled SDE

$$dX^u(t) = \alpha(t, X^u_t, u(t))dt + \sigma(t, X^u_t, u(t))dB(t),$$

where the non-anticipative functionals (α, σ) satisfy standard Lipschitz conditions. Let $\mathcal{X} = ((X^k)_{k\geq 1}, \mathscr{D})$ be the Euler-type controlled imbedded discrete structure associated with X. Then,

$$\sup_{\phi \in U_0^{k,e(k,T)}} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_\infty \lesssim \epsilon_k^{\frac{1}{2}^-} \to 0, \tag{12}$$

as $k \to \infty$.

Concrete cases

Proposition Leão, O-A (2024)

Let X^u be the controlled SDE

$$dX^u(t) = \alpha(t, X^u_t, u(t))dt + \sigma dB_H(t),$$

where B_H ia a real-valued FBM with $0 < H < \frac{1}{2}$, the non-anticipative functional α satisfies standard Lipchitz assumptions. Let $\mathscr{X} = ((X^k)_{k\geq 1}, \mathscr{D})$ be the Euler-type controlled imbedded discrete structure associated with X. Then,

$$\sup_{\phi\in U_0^{k,e(k,T)}} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_\infty \lesssim \epsilon_k^{H^-} \to 0, \tag{13}$$

as $k \to +\infty$.

Concrete cases

Proposition Leão, O-A (2024)

Fix $0 < H < \frac{1}{2}$. Let X^u be the controlled SDE

$$\begin{cases} dX^u(t) = X^u(t)\mu(u(t))dt + X^u(t)\vartheta(Z(t), u(t))dB^1(t) \\ dZ(t) = \nu dW_H(t) - \beta(Z(t) - m)dt, \end{cases}$$

where $m \in \mathbb{R}, \beta, \nu > 0$, ϑ, μ satisfy standard Lipschitz assumptions and W_H is a FBM correlated to B^1. Let $\mathcal{X} = ((X^k)_{k\geq 1}, \mathscr{D})$ be the Euler-type controlled imbedded discrete structure associated with X. Then,

$$\sup_{\phi \in U_0^{k,e(k,T)}} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_\infty \lesssim \epsilon_k^{H^-} \to 0,$$

as $k \to +\infty$.

Optimal control of drifts

Theorem Leão, O-A (2024)

Let X be the controlled SDEs driven by FBM with $0 < H < \frac{1}{2}$, where the controls affect only the drift coefficients. Assume the drift has convex range in $\mathbb{R}^n$. Then, for

$$\left| \sup_{\phi \in U_0^T} \mathbb{E}[\xi_X(\phi)] - V_k(0) \right| \lesssim \epsilon_k^{H^-} \to 0, \text{ as } k \to \infty.$$

Theorem Leão, O-A (2024)

If the controlled process is a path-dependent SDE driven by a Brownian motion, then the near optimal controls are **closed-loop**. If the controlled process is a path-dependent SDE driven by a fractional Brownian motion, then the near optimal controls are **open-loop**.

Numerical example: Markovian case

For a given $c \in \mathbb{R}$ and a Lipschitz function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, we define $\varrho_c(x, y, z) := (c + x - \varphi(y, z))^2; (x, y, z) \in \mathbb{R}^3$. Let us consider

$$dS^1(t) = S^1(t)\left(\mu_1 dt + \sigma_1 dB^1(t)\right)$$
$$dS^2(t) = S^1(t)\left(\mu_2 dt + \sigma_2 dB^2(t)\right).$$

The problem is

$$\text{minimize} \quad \mathbb{E}\Big[\varrho_c(X(T,\phi), S^1(T), S^2(T))\Big] \quad \text{over all } \phi \in U_0^T,\ c \in \mathbb{R},$$

where

$$X(t,\phi) = \sum_{j=1}^{2} \int_0^t \phi_j(r) dS^j(r); \phi \in U_0^T, 0 \le t \le T.$$

Numerical example: Markovian example

In this example, we choose $\varphi(y,z) := \max(y-z,0)$ and $\bar{a}=1$. It is well-known there exists a unique choice of $(c^*,\phi^*) \in \mathbb{R} \times U_0^T$ such that

$$\inf_{(c,\phi)\in\mathbb{R}\times U_0^T} \mathbb{E}\Big[\varrho_c\big(X(T,\phi),S^1(T),S^2(T)\big)\Big]$$

$$= \mathbb{E}\Big[\varrho_{c^*}\big(X(T,\phi^*),S^1(T),S^2(T)\big)\Big] = 0,$$

where $c^* = S_0^1\Phi(d_1) - S_0^2\Phi(d_2)$,

$$\sigma = \sqrt{\sigma_1^2+\sigma_2^2}, \quad d_1 = \frac{\log\left(\frac{S^1(0)}{S^2(0)}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

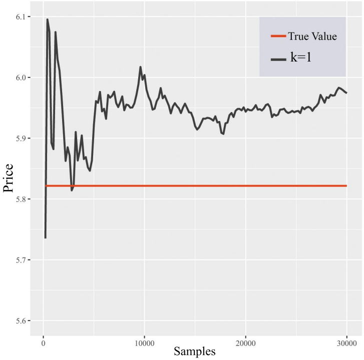
and Φ is the cumulative distribution function of the standard Gaussian variable. We recall ϕ^* is the so-called delta hedging which can be computed by means of the classical PDE Black-Scholes.

Numerical example: Mean variance hedging

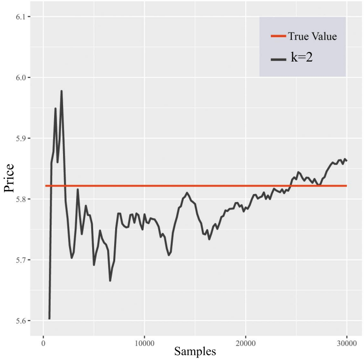
Table: Comparison between c^* and $c^{k,*}$ for $\epsilon_k = 2^{-k}$

k	Result	Mean Square Error	True Value	Difference	% Error
1	5.9740	0.01689567	5.821608	0.152458	0.0261%
2	5.8622	0.01158859	5.821608	0.04059157	0.0069%
3	5.7871	0.00821813	5.821608	0.03441365	0.0059%

Numerical example: Mean variance hedging



(a)



(b)

Figure: Monte Carlo experiments for $c^{k,*}$

Solving the dynamic programming equation by Deep Learning

The risky asset price

$$S_i^k = S(0) + \sum_{j=1}^{i} \Delta S_j^k ; 1 \leq i \leq m,$$

follows a geometric Brownian motion-type process

$$\Delta S_\ell^k = \mu S_{\ell-1}^k \Delta T_{\ell-1}^k + \sigma S_{\ell-1}^k \Delta A_\ell^k,$$

for $1 \leq \ell \leq m$. For a given control $\phi = (\phi_0, \ldots, \phi_{m-1})$, we consider the wealth process

$$Y_i^{k,\phi} = c^\star + \sum_{j=1}^{i} \phi_{j-1} \Delta S_j^k ; 1 \leq i \leq m.$$

where c The two-dimensional controlled process is $X_i^{k,\phi} := \begin{pmatrix} S_i^k \\ Y_i^{k,\phi} \end{pmatrix}$

for $i = 0, \ldots, m$.

Solving by Deep Learning

The goal is to compute

$$\phi \in \arg\min_{\phi \in U} \mathbb{E}\left| Y_m^{\phi} - \varphi(S_m^k) \right|^2$$

over a suitable class of controls U parameterized by a Feedforward Neural Network.

Solving by Deep Learning

In general, the transition function $\mathcal{X} : \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}) \times \mathbb{W} \to \mathbb{R}^2$ at the $j+1$-th step (viewed backwards for $j = m-1, \ldots, 0$) is given by

$$\mathcal{X}(\theta, x_j, w) := \begin{pmatrix} \mu x_j^{(1)} s + \sigma x_j^{(1)} \tilde{i} \\ \mu x_j^{(1)} \theta s + \sigma x_j^{(1)} \theta \tilde{i} \end{pmatrix} \tag{14}$$

where $w = (s, \tilde{i})$ for $s \in (0, \infty)$ and $\tilde{i} \in \{-2^{-k}, +2^k\}$. Then, we define recursively

$$\begin{aligned} \mathbf{U}_j(x_j, \theta) &:= \int_{\mathbb{W}} \mathbb{V}_{j+1}\Big(x_j + \mathfrak{X}(\theta, x_j, \omega)\Big) \nu(d\omega) \\ \mathbb{V}_j(x_j) &:= \inf_{\theta \in \mathbb{R}} \mathbf{U}_j(x_j, \theta), \end{aligned} \tag{15}$$

for $j = m-1, \ldots, 0$.

We define u_j^{po} as follows:

$$u_j^{op} \in \arg\min_{\theta \in \mathbb{R}} \mathbf{U}_j(x_j, \theta),$$

for $j = m-1, \ldots, 0$. Observe that this iterative scheme defines a sequence of Borel functions $g_j : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}; j = m-1, \ldots, 0$ which realizes

$$u_j^{op}(x_j) = g_j(x_j),$$

for each $x_j \in \mathbb{R}_+ \times \mathbb{R}$ and $j = m-1, \ldots, 0$.

Solving by Deep Learning

By definition, the value functions are

$$\mathbb{V}_j(x_j) = \mathbf{U}_j(x_j, g_j(x_j)); j = m-1, \ldots, 0.$$

The class of functions which contains $\{g_j\}_{j=0}^{m-1}$ is unknown. For this reason, we postulate two neural network spaces (here $\mathcal{B}$ and Θ are suitable parameter sets).

$$\mathcal{C} = \{\mathbb{R}_+ \times \mathbb{R} \ni x \mapsto C(x, \theta) \in \mathbb{R}^2; \theta \in \Theta\} \tag{16}$$

and

$$\mathcal{V} = \{\mathbb{R}_+ \times \mathbb{R} \ni x \mapsto \Phi(x, \beta) \in \mathbb{R}^3; \beta \in \mathcal{B}\}. \tag{17}$$

Construction of a synthetic training data

We generate $\phi_0, \dots, \phi_{m-1}$ following uniform distributions in $[-2, 2]$. Starting with $(S_0^k, Y_0^k) = (S_0, c^\star)$, we construct

$$Y_i^{k,\phi} = c^\star + \sum_{j=1}^{i} \phi_{j-1} \Delta S_j^k ; 1 \leq i \leq m. \tag{18}$$

Solving by Deep Learning

For a given $C = (a, b) \in \mathcal{C}, \Phi = (c, d, e) \in \mathcal{V}$ and a training data $X_{j-1}^k := X_{j-1}^{k,\phi}$, we define

$$\mathcal{X}_{j-1}^{\theta} := \mathcal{X}(\widetilde{C}(X_{j-1}^k, \theta), X_{j-1}^k, \Delta T_1^k, \Delta A^k(T_1^k)), \qquad (19)$$

where

$$\widetilde{C}(X_{j-1}^k, \theta) := a(S_{j-1}^k; \theta) + b(S_{j-1}^k; \theta) Y_{j-1}^k, \qquad (20)$$

for $1 \leq j \leq m$, $\theta \in \Theta$, and

$$\widetilde{\Phi}(X_{j-1}^k, \beta) := c(S_{j-1}^k; \beta) + d(S_{j-1}^k; \beta) Y_{j-1}^k + e(S_{j-1}^k; \beta) (Y_{j-1}^k)^2, \qquad (21)$$

for $1 \leq j \leq m$ and $\beta \in \mathcal{B}$. Here, e is non-negative to insure convexity.

Solving by Deep Learning

Terminal condition: $\widehat{\mathbb{V}}_m := \mathbb{V}_m$,

1. Compute the approximated control at time n

$$\overbrace{\hat{\theta}_n \in \operatorname*{arg\,min}_{\theta \in \Theta}}^{\text{stochastic gradient descent(ADAM)}} \quad \mathbb{E}\left[\widehat{\mathbb{V}}_{n+1}\left(X_n^k + \mathcal{X}_n^{\theta}\right)\right] \tag{22}$$

2. compute the estimation of the value function at time n

$$\overbrace{\widehat{\mathbb{V}}_n \in \operatorname*{arg\,min}_{\beta \in \mathcal{B}}}^{\text{stochastic gradient descent (ADAM)}} \quad \mathbb{E}\left[\widehat{\mathbb{V}}_{n+1}(X_n^k + \mathcal{X}_n^{\hat{\theta}_n}) - \widetilde{\Phi}(X_n^k, \beta)\right]^2 \tag{23}$$

for $n = m - 1, \ldots, 1, 0$.

Numerical example: Mean variance hedging

Table: Computing the Profit and Loss by Deep Learning

k	mean	Standard Deviation
1	0.3740	0.1689567
2	0.1622	0.4158859
3	0.02871	0.10821813

THANK YOU VERY MUCH FOR YOUR ATTENTION !!