Existence and uniqueness results for strongly degenerate McKean-Vlasov equations with rough coefficients

Alessio Rondelli
Joint work with A. Pascucci and A. Y. Veretennikov

25 June 2025

Weak Existance

Theorem

Consider

$$\begin{cases} \mathrm{d}X_{0,t} = B_0(t, X_t, [X_t])\mathrm{d}t \\ \mathrm{d}X_{1,t} = B_1(t, X_t, [X_t])\mathrm{d}t + \Sigma_1(t, X_t, [X_t])\mathrm{d}W_t^x, \qquad X_0 = \check{X}_0, \end{cases}$$

where B_0, B_1, Σ_1 are of the type

$$\Sigma_1(t, X_t, [X_t]) = \mathbb{E}\left[\sigma_1(t, x, X_t)\right]_{|x=X_t} = \int_{\mathbb{R}^N} \sigma_1(t, X_t, x)[X_t](\mathrm{d}x).$$

Weak Existance

$$\begin{cases} \mathrm{d}X_{0,t} = B_0(t, X_t, [X_t])\mathrm{d}t \\ \mathrm{d}X_{1,t} = B_1(t, X_t, [X_t])\mathrm{d}t + \Sigma_1(t, X_t, [X_t])\mathrm{d}W_t^x, \qquad X_0 = \check{X}_0, \end{cases}$$

Theorem

Under the hypotheses

- σ_1 *is a* $d \times d$ *uniformly non degenerate matrix*
- b *and* σ *of sublinear growth in* x *and* y *uniformly in* t
- *The initial datum* $\check{X}_0 \in L^{2+\epsilon}$.
- $b(t, x_0, x_1, y_0, y_1)$ *and* $\sigma_1(t, x_0, x_1, y_0, y_1)$ *continuous in the degenerate variables uniformly wrt time and the non degenerate variables*

exists a weak solution to the SDE.

Remark

- *This result covers both ODEs and non degenerate SDEs with minimal hypotheses in both cases:*

Remark

- *This result covers both ODEs and non degenerate SDEs with minimal hypotheses in both cases:*
    - *For ODEs the coefficients must be at least continuous to have existance results.*

Remark

- *This result covers both ODEs and non degenerate SDEs with minimal hypotheses in both cases:*
  - *For ODEs the coefficients must be at least continuous to have existance results.*
  - *For SDEs measurability is enough due to the regularization by noise effect of the Brownian motion. (Controlled Diffusion Processes, Krylov* **1980***)*

Remark

- *This result covers both ODEs and non degenerate SDEs with minimal hypotheses in both cases:*
  - *For ODEs the coefficients must be at least continuous to have existance results.*
  - *For SDEs measurability is enough due to the regularization by noise effect of the Brownian motion. (Controlled Diffusion Processes, Krylov* **1980***)*
- *For applications it is of utmost importance that the coefficients may be very rough.*

Classical Result for MKV SDEs

Theorem

Given a MKV SDE with Lipschitz coefficients of sublinear growth and L^p initial datum η we have strong well-posedness. Also the following estimates are true

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t|^p\right] \le C(1+||\eta||^p_{L^p}),$$

$$\sup_{|t-s|\le h}\mathbb{E}\left[|X_t-X_s|^p\right] \le Ch^{\frac{p}{2}},$$

where C only depends on the sublinear growth coefficient and the dimension of the problem.

Proof

Regularization: Firstly we truncate and regularize the coefficients to construct a sequence of strong solutions to the regularized SDEs

$$B^n(t, X_t, [X_t]),\ \Sigma^n(t, X_t, [X_t]) \qquad \Rightarrow \qquad X^n$$

these solutions exist due to the classical result and the fact the the regularized coefficients are Lipschitz.
The estimates of the classical result are even uniform in n.

Proof

Linearization: Due to the classical theorem we can even find a sequence Y^n of processes **equal in law** to X^n but **independent**, this way the SDE becomes

$$\begin{cases} \mathrm{d}X^n_{0,t} = B^n_0(t, X^n_t, [Y^n_t])\mathrm{d}t \\ \mathrm{d}X^n_{1,t} = B^n_1(t, X^n_t, [Y^n_t])\mathrm{d}t + \Sigma^n_1(t, X^n_t, [Y^n_t])\mathrm{d}W^{x,n}_t, \end{cases}$$

at this point this new SDE can be seen as classic since the law of the solution doesn't appear in the coefficients.

Proof

Skorokhod's Lemma: We want to send the SDE to the limit, to do that we use Skorokhod's Lemma that can be thought as a probabilistic version of Ascoli-Arzelà's theorem.

Since these two estimates are verified (by the uniform estimates)

- $\lim_{\lambda \to 0} \sup_n \sup_t P(|X_t^n| > \lambda) = 0$, (**equiboundedness**)
- $\lim_{h \to 0} \sup_n \sup_{\substack{t,s \in [0,T] \\ |t-s| \leq h}} P(|X_t^n - X_s^n| > \epsilon) = 0$, (**equicontinuity**)

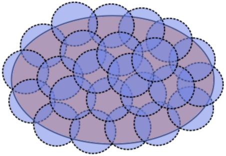
we can copy the processes in a new probability space where

$$(\tilde{X}_t^n, \tilde{Y}_t^n) \xrightarrow{P} (\tilde{X}_t^\infty, \tilde{Y}_t^\infty), \quad n \to \infty, \quad \forall t \in [0, T].$$

Then it is possible to prove that the uniform estimates still work for the copied processes and that $\tilde{X}^n$ is a solution to the same SDE as X^n.

Proof

ϵ-net: To manipulate the degenerate components in the limit we need to build an ϵ-net over the bounded Hölder functions. Since it is possible to prove that "most" of the trajectories of the degenerate components are uniformly Hölder and bounded there exists a finite amount of functions ϕ_j that are close to these components. With these we can approximate most of the $X^n_{0,t}$ and $Y^n_{0,t}$.



Theorem (Krylov's Bounds)

Let Z_t be a stochastic process over $\mathbb{R}^d$ such that

$$\mathrm{d}Z_t = b_t(Z_t)\mathrm{d}t + \sigma_t(Z_t)\mathrm{d}W_t, \qquad Z_0 = Z_0,$$

where the coefficients b, σ are measurable with σ uniformly non degenerate. Let D be a bounded set. Then for any $p \geq d$ such that for any $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ measurable that vanishes outside D

$$\mathbb{E}\left[\int_0^T |f(t, Z_t)|\mathrm{d}t\right] \leq N||f||_{L^{p+1}([0,T]\times D)},$$

where the constant N that depends only on the space dimension, on D and on the ellipticity constant of $\sigma\sigma^*$.

Proof

With these instruments we are able to finish the proof: indeed we want to prove convergence of the coefficients:

$$\int_0^t \mathbb{E}\left[\left|B^n(s, X_s^n, [Y_s^n]) - B(s, X_s^\infty, [Y_s^\infty])\right|\right] ds$$

Proof

With these instruments we are able to finish the proof: indeed we want to prove convergence of the coefficients:

$$\int_0^t \mathbb{E}\left[|B^n(s, X_s^n, [Y_s^n]) - B(s, X_s^\infty, [Y_s^\infty])|\right] ds$$
$$\leq \int_0^t \mathbb{E}\left[\mathbb{E}\left[|b^n(s, x, Y_s^n) - b(s, z, Y_s^\infty)|\right]|_{x=X_s^n, z=X_s^\infty}\right] ds$$

Proof

With these instruments we are able to finish the proof: indeed we want to prove convergence of the coefficients:

$$\int_0^t \mathbb{E}\left[\left|B^n(s, X_s^n, [Y_s^n]) - B(s, X_s^\infty, [Y_s^\infty])\right|\right] ds$$

$$\leq \int_0^t \mathbb{E}\left[\mathbb{E}\left[\left|b^n(s, x, Y_s^n) - b(s, z, Y_s^\infty)\right|\right]\Big|_{x=X_s^n, z=X_s^\infty}\right] ds$$

$$\overset{Freezing}{\approx} \int_0^t \mathbb{E}\left[\left|b^n(s, X_s^n, Y_s^n) - b(s, X_s^\infty, Y_s^\infty)\right|\right] ds$$

Proof

With these instruments we are able to finish the proof: indeed we want to prove convergence of the coefficients:

$$\int_0^t \mathbb{E}\left[|B^n(s, X_s^n, [Y_s^n]) - B(s, X_s^\infty, [Y_s^\infty])|\right] ds$$

$$\leq \int_0^t \mathbb{E}\left[\mathbb{E}\left[|b^n(s, x, Y_s^n) - b(s, z, Y_s^\infty)|\right]|_{x=X_s^n, z=X_s^\infty}\right] ds$$

$$\overset{Freezing}{\approx} \int_0^t \mathbb{E}\left[|b^n(s, X_s^n, Y_s^n) - b(s, X_s^\infty, Y_s^\infty)|\right] ds$$

$$\approx \int_0^t \mathbb{E}\left[\left|b^n(s, \phi_s^k, X_{1,s}^n, \phi_s^j, Y_{1,s}^n) - b(s, \phi_s^k, X_{1,s}^\infty, \phi_s^j, Y_{1,s}^\infty)\right|\right]$$

Proof

With these instruments we are able to finish the proof: indeed we want to prove convergence of the coefficients:

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left[\left|B^n(s, X_s^n, [Y_s^n]) - B(s, X_s^\infty, [Y_s^\infty])\right|\right] ds \\
&\leq \int_0^t \mathbb{E}\left[\mathbb{E}\left[\left|b^n(s, x, Y_s^n) - b(s, z, Y_s^\infty)\right|\right]\big|_{x=X_s^n, z=X_s^\infty}\right] ds \\
&\overset{Freezing}{\approx} \int_0^t \mathbb{E}\left[\left|b^n(s, X_s^n, Y_s^n) - b(s, X_s^\infty, Y_s^\infty)\right|\right] ds \\
&\approx \int_0^t \mathbb{E}\left[\left|b^n(s, \phi_s^k, X_{1,s}^n, \phi_s^j, Y_{1,s}^n) - b(s, \phi_s^k, X_{1,s}^\infty, \phi_s^j, Y_{1,s}^\infty)\right|\right] \\
&\overset{Krylov}{\leq} N||b^n - b||_{L^{2d+1}} \to 0.
\end{aligned}
$$

Uniqueness

Theorem

Consider

$$\begin{cases} \mathrm{d}X_{0,t} = B_0(t, X_t)\mathrm{d}t \\ \mathrm{d}X_{1,t} = B_1(t, X_t, [X_t])\mathrm{d}t + \Sigma_1(t, X_t)\mathrm{d}W_t^x, \end{cases} \qquad X_0 = \eta,$$

where B_1 is of the type

$$B_1(t, X_t, [X_t]) = \mathbb{E}\left[b_1(t, x, X_t)\right]_{|x=X_t} = \int_{\mathbb{R}^N} b_1(t, X_t, x)[X_t](\mathrm{d}x).$$

Uniqueness

suppose

- $b(t, x, y)$ is of sublinear growth in x uniformly in (t, y).
- σ_1 bounded and uniformly definite positive
- η satisfies $\mathbb{E}[e^{r\eta^2}] < \infty$ for some $r > 0$.

then if we have weak (resp. strong) uniqueness of the linearized SDE for every flow of marginals we have also weak (resp. strong) uniqueness of the MKV SDE.

Remark

- *If b_1 is bounded the initial datum just needs to be in L^2.*

Remark

- *If b_1 is bounded the initial datum just needs to be in L^2.*
- *While the theorem doesn't require regularity of the coefficients the assumption about weak (or strong) well-posedness of the linearized SDE forces some regularity on the coefficients, some known results are that weak well-posedness is achieved with bounded and intrinsic Hölder coefficients while for strong well-posedness the Hölder coefficients also need to meet certain thresholds.*

Remark

- *If b_1 is bounded the initial datum just needs to be in L^2.*
- *While the theorem doesn't require regularity of the coefficients the assumption about weak (or strong) well-posedness of the linearized SDE forces some regularity on the coefficients, some known results are that weak well-posedness is achieved with bounded and intrinsic Hölder coefficients while for strong well-posedness the Hölder coefficients also need to meet certain thresholds.*
- *The proof is based on Girsanov's theorem and thus the diffusion coefficient needs to be independent of the law.*

Thanks for your attention!

Related Literature

[1] Chaudru de Raynal, *P. E. Strong well posedness of McKean-Vlasov stochastic differential equations with Hölder drift*. Stochastic Process. Appl. 130, 1 (2020), 79–107.

[2] Chaudru de Raynal, P.-E., and Frikha, N. *Well-posedness for some non-linear SDEs and related PDE on the Wasserstein space*. J. Math. Pures Appl. (9) 159 (2022), 1–167.

[3] Hao, Z., Röckner, M., and Zhang, X. *Strong convergence of propagation of chaos for McKean- Vlasov SDEs with singular interactions*. SIAM J. Math. Anal. 56, 2 (2024), 2661–2713.

[4] Issoglio, E., Pagliarani, S., Russo, F., and Trevisani, D. *Degenerate McKean-Vlasov equations with drift in anisotropic negative Besov spaces*. Preprint arXiv:2401.09165 (2024).

[5] Mishura, Y., and Veretennikov, A. Y. *Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations*. Theory Probab. Math. Statist., 103 (2020), 59–101.

[6] Pascucci, A., and Rondelli, A. *McKean-Vlasov stochastic equations with Hölder coefficients*. Stochastic Process. Appl. 182, 104564 (2025).

[7] Pascucci, A., Rondelli, A., and Veretennikov, A. Y. *Existence and uniqueness results for strongly degenerate McKean-Vlasov equations with rough coefficients*. Preprint arXiv: 2409.14451 (2024).

[8] Röckner, M., and Zhang, X. *Well-posedness of distribution dependent SDEs with singular drifts*. Bernoulli 27, 2 (2021), 1131–1158.

[9] A. Yu. Veretennikov (*On weak existence of solutions of degenerate McKean-Vlasov equations*, 2023)

[10] Zhao, G. *Existence and uniqueness for McKean-Vlasov equations with singular interactions*. Potential Anal. (2024).