Regularisation by fractional noise: density of SDEs and McKean-Vlasov equations

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Overview

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Consider the equation

$$\boxed{\mathrm{d}X_t = \varphi(t, X_t)\,\mathrm{d}t + \mathrm{d}B_t,} \tag{E}$$

where $\varphi(t, \cdot)$ is a distribution in some Besov space and B is a fractional Brownian motion.

We look for solutions of the form

$$X_t = X_0 + K_t + B_t,$$

where in case φ is regular enough, $K_t = \int_0^t \varphi(r, X_r)\,\mathrm{d}r$.

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Typical examples

- $\varphi = \alpha\,\delta_0$: corresponds formally to an SDE involving the local time of the solution, see [Le Gall'84] in the Brownian case.
- $\varphi = \alpha\,|\cdot|^{-s}$: Bessel-like processes and Riesz-type kernels in mathematical physics (e.g. Coulomb gases, Keller-Segel model, etc.).

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Without noise, classical theory requires

- $\varphi \in L^1_t \mathcal{C}^1_b$ for well-posedness;
- $\varphi \in L^1_t \mathcal{C}^0_b$ for mere existence (Peano).

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$$\mathrm{d}X_t = \mathsf{sign}(X_t)\sqrt{|X_t|}\,\mathrm{d}t \qquad , \quad X_0 = 0,$$

whose solutions are given, for any $t^* \in \mathbb{R}_+$, by

$$(X^{t^*}_t)_{t \in \mathbb{R}_+} := t \mapsto (t - t^*)^2_+.$$

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Without noise, classical theory requires

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Now add noise to the equation. Due to the forcing, solution leaves 0 immediately. But away from 0, Lipschitz drift $\implies$ uniqueness. For almost each trajectory of $(B_t)_{t\geq 0}$, we have a unique solution.

How is the noise helping?

Heuristics – In situations where the ODE $\dot{x}_t = \varphi(x_t)$ lacks uniqueness, adding noise might restore uniqueness $\rightsquigarrow$ regularisation by noise.

Consider $\widetilde{X} = X - B$ which now solves the random ODE:

$$\widetilde{X}_t = X_0 + \int_0^t \varphi(\widetilde{X}_r + B_r)\,\mathrm{d}r, \quad t \geq 0.$$

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In $X = \widetilde{X} + B$, $\widetilde{X}$ gives slow oscillations and B fast oscillations. Freezing $\widetilde{X}$, consider

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In fact, for L the local time of B,

$$\int_0^t \varphi(x + B_r)\,\mathrm{d}r = \int_{\mathbb{R}} \varphi(x + y)\, L_t(y)\,\mathrm{d}y = \varphi * \check{L}_t(x).$$

$\implies$ $\varphi * \check{L}_t$ is more regular than φ !

Rougher noise, smoother local time

For a Hurst parameter $H \in (0,1) \setminus \{\frac{1}{2}\}$, fractional Brownian motion (fBm) is given by:

$$B_t = c_H \int_{\mathbb{R}} \Big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \Big) \, \mathrm{d}W_s, \quad t \in \mathbb{R}.$$

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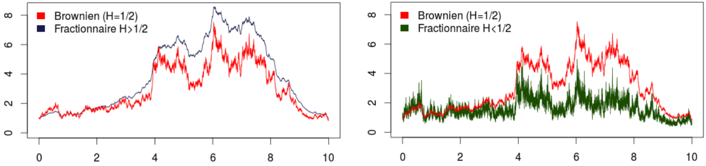

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  - $H > \frac{1}{2}$: more regular than Bm, long-range dependence.
  - Rough regime $H < \frac{1}{2}$: negatively correlated increments, strong oscillations.

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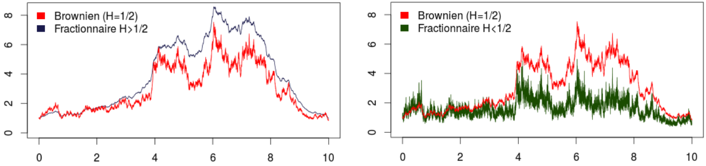

- Gaussian process with memory:
  - $H > \frac{1}{2}$: more regular than Bm, long-range dependence.
  - Rough regime $H < \frac{1}{2}$: negatively correlated increments, strong oscillations.
- Local time: $x \mapsto L_t(x)$ has regularity $\frac{1}{2H} - \frac{1}{2} - \varepsilon$ a.s.

Rule of thumb: rougher noise, better regularisation!

A few results - Brownian case

- Works of Zvonkin, Veretennikov, [Krylov & Röckner'05]: Strong WP for $\varphi(t,x) \in L^q([0,T];L^p(\mathbb{R}^d))$

$$\text{if } \; p \geq 2, \; q > 2, \; \frac{2}{q} + \frac{d}{p} < 1.$$

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- Hölder setting: $\varphi \in \mathcal{C}^\gamma$,
  - [Bass & Chen'01] proved strong WP for $\gamma > -\frac{1}{2}$, $d = 1$, counter-examples for $\gamma < -\frac{1}{2}$.
  - Weak WP for $\gamma > -\frac{2}{3}$, $d = 1$ [Delarue & Diel'16]; weak WP for $\gamma > -\frac{1}{2}$, $d \geq 1$ [Flandoli, Issoglio & Russo'17]; Canizzaro-Chouk, Coutin-Duboscq-Réveillac, etc.
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These results rely crucially on the Markov property of the BM, and subsequently on PDE techniques (martingale problem and/or Zvonkin transform).

A few results - fBm case

But fBm is neither Markov, nor a semimartingale.

- Early work by [Nualart & Ouknine'02]. Then [Catellier & Gubinelli'16] used *nonlinear Young integration* to prove that there is a unique solution if

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- Recently, thanks to the Stochastic Sewing Lemma of [Lê'20],
  - For $\varphi \in \mathcal{B}_p^\gamma(\mathbb{R}^d)$, $p < \infty$, $\gamma - \frac{d}{p} = 1 - \frac{1}{2H}$, strong WP of the fBm-driven SDE [Anzeletti, R. & Tanré'23];
  - Weak well-posedness: weak existence in [Anzeletti, R. & Tanré'23] for $\gamma > \frac{1}{2} - \frac{1}{2H}$, uniqueness in law in [Butkovsky & Mytnik '24].

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Theorem ([Galeati & Gerencsér'24] – *Time-dependent* drift)

Strong WP holds for (E) *when* $\varphi \in L^q([0,T]; \mathcal{C}^\gamma(\mathbb{R}^d))$ *with*
$$\gamma > 1 - \frac{1}{H(q' \vee 2)} \quad \textit{and} \quad q \in (1, \infty].$$

McKean-Vlasov equations

As for "linear" SDEs, it is possible to exploit the regularising effect of the noise for McKean-Vlasov SDEs. Consider specifically convolution-type equations

$$\begin{cases} \mathrm{d}Y_t = \psi_t * \mu_t(Y_t)\,\mathrm{d}t + \mathrm{d}B_t \\ \mu_t = \mathsf{Law}(Y_t). \end{cases} \tag{McKV}$$

This eq. arises formally as the limit of interacting particle systems.

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Theorem ([Galeati, Harang & Mayorcas'23], [Galeati & Gerencsér'24])

Strong WP holds for (McK-V) *when* $\psi \in L^q \mathcal{C}^\gamma$ *with* $\gamma > 1 - \frac{1}{H(q' \vee 2)}$ *and* $q \in (1, \infty]$.

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Remark: a similar statement holds for more general drift $\Psi(t, x, \mu)$.

Objectives:

- Obtain the regularity of the law of a *linear* SDE;
- Exploit this regularity for (McK-V) to go below the $1 - \frac{1}{H(q' \vee 2)}$ threshold.

Overview

Introduction

Regularity of laws of SDEs

Gaussian bounds

McKean-Vlasov equations

- For $\varphi \in L^\infty([0,T];\mathcal{C}^\gamma(\mathbb{R}^d))$ and $H > \frac{1}{2}$, [Olivera & Tudor'19] : X_t has a density with some Besov regularity.

- For $\varphi \in L^\infty([0,T];\mathcal{C}^\gamma(\mathbb{R}^d))$, [Galeati, Harang & Mayorcas'23] show that $\mathcal{L}(X_\cdot) \in L^{\tilde{q}}([0,T];\mathcal{B}_1^\alpha)$ for $\alpha < \frac{1}{H}(\frac{1}{\tilde{q}} - \frac{1}{2})$.

Besov regularity

$$\mathcal{B}_1^{\alpha}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\mathcal{B}_1^{\alpha}} < \infty \right\},$$

where $\|\cdot\|_{\mathcal{B}_1^{\alpha}}$ has the equivalent thermic representation:

$$\|\mathcal{F}^{-1}\left(\phi \mathcal{F} f\right)\|_{L^1(\mathbb{R}^d)} + \sup_{s\in(0,1]} s^{\frac{n-\alpha}{2}} \|\partial_s^n g_s * f\|_{L^1(\mathbb{R}^d)},$$

for any $n \geq \alpha$, $n \in \mathbb{N}$.

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for any $n \geq \alpha$, $n \in \mathbb{N}$.

For instance, one gets for the fBm B of Hurst parameter $H \in (0,1)$ that

$$\|\mathcal{L}(B_t)\|_{\mathcal{B}_1^\alpha} = \|g_{t^{2H}}\|_{\mathcal{B}_1^\alpha} \lesssim \frac{1}{1 \wedge t^{\alpha H}}, \ \forall t > 0.$$

In particular, $\mathcal{L}(B_\cdot) \in L^{\widetilde{q}}([0,T]; \mathcal{B}_1^\alpha)$ when $\alpha < \frac{1}{H\widetilde{q}}$.

$$X_t = X_0 + \int_0^t \varphi(s, X_s)\,\mathrm{d}s + B_t, \quad t \in [0,T]. \tag{E}$$

Definition

- *Solution*:
  - $(\varphi^n)_{n\in\mathbb{N}}$ in $L^q([0,T];\mathcal{C}_b^\infty)$, $\varphi^n \to \varphi$ in $L^q([0,T];\mathcal{C}^{\gamma-})$.
  - $\forall n \in \mathbb{N}$, denote X^n the solution of (E) with drift φ^n.
  - If $(X^n)_{n\in\mathbb{N}}$ converges in $L^2(\Omega;\mathcal{C}_{[0,T]})$, call the limit $(X_t)_{t\in[0,T]}$ a *solution* to (E).

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Assumption: $\varphi \in L^q([0,T];\mathcal{C}^\gamma(\mathbb{R}^d))$ with

$$H \in (0,+\infty)\setminus\mathbb{N}, \quad \gamma > 1 - \frac{1}{H(q' \vee 2)} \text{ and } q \in (1,+\infty]. \tag{A}$$

Time-space regularity of the density

Theorem (Anzeletti, Galeati, R. & Tanré '25)

Under (A), *let* X *be the solution to* (E). *Let* $\tilde{q} \in [1, \infty)$ *and*

$$0 \le \alpha < \min\left\{ \frac{1}{H\widetilde{q}}, \gamma - 1 + \frac{1}{H} \right\}.$$

Then for any $0 \le s < t \le T$, $(\gamma \le 0,)$

$$\|\mathcal{L}(X_\cdot)\|_{L^{\widetilde{q}}_{[s,t]}\mathcal{B}^\alpha_1} \lesssim (t-s)^{\frac{1}{\widetilde{q}} - \alpha H} + \left(\|\varphi\|_{L^q_{[s,t]}\mathcal{C}^\gamma} + \|\varphi\|^{1+\eta}_{L^q_{[s,t]}\mathcal{C}^\gamma}\right)(t-s)^\varepsilon,$$

where

$$\varepsilon = \frac{1}{q'} + \frac{1}{\widetilde{q}} - H(\alpha+1) + \min\left(-\frac{\eta}{q}, \gamma H\right) > 0$$

and

$$\eta = \frac{-\gamma H}{1 + H\gamma - H} \in (0,1).$$

Theorem (More general version)

Under (A), *let* X *be the solution to* (E) *starting from an* $\mathcal{F}_0$*-measurable random variable* X_0.

(a) *For*

$$0 < \alpha < \gamma - 1 + \frac{1}{Hq'},$$

then for any $0 \le u < t \le T$, *the conditional law* $\mathcal{L}(X_t \mid \mathcal{F}_u)$ *has a density which satisfies*

$$\Big\| \|\mathcal{L}(X_t \mid \mathcal{F}_u)\|_{\mathcal{B}_1^\alpha} \Big\|_{L^\infty_\Omega} \leqslant C(1 + (t-u)^{-\alpha H}).$$

(b) *Let* $(\tilde{q}, \alpha)$ *satisfying*

$$\tilde{q} \in (1, +\infty), \qquad 0 < \alpha < \min\left\{\frac{1}{H\tilde{q}}, \gamma - 1 + \frac{1}{H}\right\},$$

then for any $u \in [0, T)$, $t \mapsto \mathcal{L}(X_t \mid \mathcal{F}_u)$ *belongs a.s. to* $L^{\tilde{q}}([u, T]; \mathcal{B}_1^\alpha)$ *and satisfies*

$$\Big\| \|\mathcal{L}(X_\cdot \mid \mathcal{F}_u)\|_{L^{\tilde{q}}([u,T];\mathcal{B}_1^\alpha)} \Big\|_{L^\infty_\Omega} \leqslant C(T-u)^{\frac{1}{\tilde{q}} - \alpha H}.$$

- For $q = \tilde{q} = 2$, the condition on γ is $\gamma > 1 - \frac{1}{2H}$ and the density estimate becomes

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for any $\alpha < \frac{1}{2H}$.

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- Similarly for $\tilde{q} = 1+$, $\mathcal{L}(X_\cdot) \in L^{1+}_{[s,t]}\mathcal{B}^\alpha_1$ for $\alpha < \frac{1}{H}$.

Scheme of proof - 1

Fix $s < t$. By a duality argument,

$$\|\mathcal{L}(X_\cdot)\|_{L^{\widetilde{q}}_{[s,t]}\mathcal{B}^\alpha_1} \lesssim \sup_{\substack{f\in L^{\widetilde{q}'}_{[s,t]}\mathcal{C}^{-\alpha},\ \|f\|\leq 1 \\ f \text{ smooth}}} \Big| \int_s^t \langle f_r, \mathcal{L}(X_r)\rangle \,\mathrm{d}r \Big|$$

$$\lesssim \sup_{\substack{f\in L^{\widetilde{q}'}_{[s,t]}\mathcal{C}^{-\alpha},\ \|f\|\leq 1 \\ f \text{ smooth}}} \Big| \mathbb{E} \int_s^t f_r(X_r)\,\mathrm{d}r \Big|.$$

The above expectation of $\int_s^t f_r(X_r)\,\mathrm{d}r$ can now be studied *via sewing techniques*.

Scheme of proof - 2

Lemma

Assume (A), $\gamma < 0$. *Let* $\tilde{q} \in [1, \infty)$ *and*

$$0 \le \alpha < \min\left\{ \frac{1}{H\widetilde{q}}, \gamma - 1 + \frac{1}{H} \right\}.$$

For any $f \in L^{\widetilde{q}'}([0,T]; \mathcal{C}_b^\infty(\mathbb{R}^d))$ *and any* $0 \le s < t \le T$,

$$\left| \mathbb{E} \int_s^t f_r(X_r)\, dr \right| \lesssim \|f\|_{L^{\widetilde{q}'}_{[s,t]} \mathcal{C}^{-\alpha}} \times \left((t-s)^{\frac{1}{\widetilde{q}} - \alpha H} + (\|\varphi\|_{L^q_{[s,t]} \mathcal{C}^\gamma} + \|\varphi\|^{1+\eta}_{L^q_{[s,t]} \mathcal{C}^\gamma})(t-s)^\varepsilon \right).$$

Overview

Gaussian bounds for the density of X

Recent results giving Gaussian bounds on the density of SDEs:

- [Besalú et al.'16], [Baudoin et al.'16] : Rough differential equations driven by fBm, smooth vector fields, using Malliavin calculus;
- [Li, Panloup & Sieber'23]: Differential equations with additive fractional noise, irregular drift *function* in the Catellier-Gubinelli regime, i.e. imposes restrictions when $H < 1/2$;
- [Perkowski & van Zuijlen'23]: upper and lower bound on the density of SDEs, distributional drift with reg. $> -\frac{1}{2}$.

Gaussian bounds

$$dX_t = \varphi(t, X_t)dt + dB_t. \tag{E}$$

For $H \geq 1/2$, $\gamma > 1 - 1/(2H)$ and $\varphi \in L^\infty([0,T]; \mathcal{C}^\gamma(\mathbb{R}^d))$,
[Li, Panloup & Sieber'23] proved upper and lower Gaussian bounds.

Theorem

Let $H \leq 1/2$, $\gamma > 1 - 1/(2H)$ and $\varphi \in L^\infty([0,T]; \mathcal{C}^\gamma(\mathbb{R}^d))$.
Then the solution to (E) *has a density for any $t \in (0,T]$ and $\exists C > 0$ s.t. $\forall t \in (0,T]$, $\forall x \in \mathbb{R}^d$,*

$$\frac{C^{-1}}{t^{dH}} \exp\left(-C\frac{|x-x_0|^2}{t^{2H}}\right) \leq \frac{d\mathcal{L}(X_t)}{dx}(x) \leq \frac{C}{t^{dH}} \exp\left(-C^{-1}\frac{|x-x_0|^2}{t^{2H}}\right).$$

Girsanov's formula for fBm gives for the density of X_1:

$$\frac{\mathrm{d}\mathcal{L}(X_1)}{\mathrm{d}y}(y) = (2\pi)^{-dH} e^{-\frac{|y|^2}{2}} \Psi(y),$$

where

$$\Psi(y) = \mathbb{E}\Big[\exp\Big(\int_0^1 (K_H^{-1} Z)_s \cdot \mathrm{d}W_s - \frac{1}{2} \int_0^1 |(K_H^{-1} Z)_s|^2 \, \mathrm{d}s \Big) \mid B_1 = y \Big],$$

K_H is a nonlocal operator from the definition of fBm, and $Z_\cdot = \int_0^\cdot \varphi(s, B_s) \, \mathrm{d}s$.

Girsanov's formula for fBm gives for the density of X_1:

$$\frac{\mathrm{d}\mathcal{L}(X_1)}{\mathrm{d}y}(y) = (2\pi)^{-dH} e^{-\frac{|y|^2}{2}} \Psi(y),$$

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- Study above exponential functionals using regularising effects (SSL+LND) of those Gaussian bridges (recall that φ can still be distributional!).

Overview

Consider convolution-type McKean-Vlasov SDEs:

$$\begin{cases} \mathrm{d}Y_t = \psi_t * \mu_t(Y_t)\,\mathrm{d}t + \mathrm{d}B_t \\ \mu_t = \mathsf{Law}(Y_t),\ t \geq 0. \end{cases} \tag{McK-V}$$

Arises formally as the limit of interacting particle systems, as $N \to +\infty$:

$$\begin{cases} \mathrm{d}Y_t^{i,N} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi_t(Y_t^{i,N} - Y_t^{j,N})\,\mathrm{d}t + \mathrm{d}B_t^i, \quad i \in \{1,\dots,N\} \\ B^1,\dots,B^N \text{ independent fBm.} \end{cases}$$

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Existence

$$\begin{cases} \mathrm{d}Y_t = \psi_t * \mu_t(Y_t)\,\mathrm{d}t + \mathrm{d}B_t \\ \mu_t = \mathrm{Law}(Y_t),\ t \geq 0. \end{cases} \qquad \text{(McK-V)}$$

Theorem

Let $\psi \in L^\infty([0,T];\mathcal{C}^\theta)$ *with*

$$\theta > 1 - \frac{1}{H}.$$

There exists Y *and a family* $(\mu_t)_{t\in[0,T]}$ *solution of* (McK-V), *i.e.:*

- *for* $\rho \in [1,\infty)$ *and* $\alpha < \frac{1}{\rho H}$, $\mu \in L^\rho([0,T];\mathcal{B}^\alpha_1)$;
- Y *is the unique strong solution of the (linear) SDE with drift* $\psi * \mu \in L^\rho([0,T];\mathcal{C}^1_b)$;
- *For any* $t \geq 0$, μ_t *is the law of* Y_t.

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  $\rightsquigarrow$ In particular in $d = 2$, $s = 1$ corresponds to Coulombian interaction. In case $H = \frac{1}{2}$ and the kernel is attractive $\equiv$ Keller-Segel model, which is known to have blow-ups in certain regimes.
- A heuristic scaling argument permits to retrieve the condition $\theta > 1 - \frac{1}{H}$.

Uniqueness

Theorem

Let $H \in (0, +\infty) \setminus \mathbb{N}$ *and* $\psi \in L^\infty([0,T]; \mathcal{B}_p^\theta)$ *for some* $\theta \in (-\infty, 1)$, $p \in [1, \infty]$ *satisfying*

$$\theta > 1 - \frac{1}{2H}, \quad \theta - \frac{d}{p} > 1 - \frac{1}{H}.$$

Further assume that $\mathcal{L}(Y_0) \in L^\infty(\mathbb{R}^d)$. *Then pathwise uniqueness and uniqueness in law hold for* (McK-V), *in the class of solutions such that* $\psi * \mu \in L^1([0,T]; \mathcal{C}_b^1)$.

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- The condition $\theta - \frac{d}{p} > 1 - \frac{1}{H}$ still permits to reach a subcritical regime, up to working in Besov spaces with $p < \infty$.

Sketch of proof (existence)

- Consider the smooth approximations

$$\begin{cases} Y^n_t = Y_0 + \int_0^t \psi^n_s * \mu^n_s(Y^n_s)\,\mathrm{d}s + B_t \\ \mu^n_t = \mathcal{L}(Y^n_t),\ t \geq 0, \end{cases}$$

  which have a pathwise unique, strong solution for any $n \in \mathbb{N}$.
- Apply the density Theorem with $q = \tilde{q} = 2$, $\gamma = \alpha + \theta \approx 1 - 1/(2H)$ which gives us the condition $\alpha < 1/(2H)$:

$$\begin{aligned} \|\mu^n\|_{L^2_{[s,t]}\mathcal{B}^\alpha_1} &\lesssim (t-s)^\varepsilon + (t-s)^\varepsilon \|\psi^n * \mu^n\|^{1+\eta}_{L^2_{[s,t]}\mathcal{C}^{\theta+\alpha}} \\ &\lesssim (t-s)^\varepsilon \left(1 + \|\psi_n\|^{1+\eta}_{L^\infty_{[0,t]}\mathcal{C}^\theta} \|\mu^n\|^{1+\eta}_{L^2_{[s,t]}\mathcal{B}^\alpha_1}\right) \end{aligned}$$

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- Proceed with Kolmogorov's tightness criterion for $(Y_n)_{n\in\mathbb{N}}$.
- Identify the limit points as solutions of the McKean-Vlasov equation.

Thank you!

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Time-dependent drift – What scaling γ?

$$dX_t = \varphi(t, X_t)dt + dB_t. \qquad \text{(E)}$$

Following the scaling argument of [Galeati & Gerencsér'24], consider $B_t^{(\lambda)} = \lambda^{-H} B_{\lambda t}$ and $\varphi^{(\lambda)}(t,x) = \lambda^{1-H}\varphi(\lambda t, \lambda^H x)$. Then $X_t^{(\lambda)} = \lambda^{-H} X_{\lambda t}$ solves

$$dX_t^{(\lambda)} = \varphi^{(\lambda)}(t, X_t^{(\lambda)})\, dt + dB_t^{(\lambda)}.$$

Now observe that

$$\|\varphi^{(\lambda)}\|_{L^q\mathcal{C}^\gamma} = \lambda^{1-H-\frac{1}{q}+\gamma H}\|\varphi\|_{L^q\mathcal{C}^\gamma}.$$

As $\lambda \to 0$, we want to keep $\lambda^{1-H-\frac{1}{q}+\gamma H}$ bounded, so heuristically,

$$\gamma > 1 - \frac{1}{Hq'}.$$

Theorem ([Galeati & Gerencsér'24])

Strong WP holds for (E) *when* $\varphi \in L^q\mathcal{C}^\gamma$ *with* $\gamma > 1 - \frac{1}{Hq'}$ *and* $q' \geq 2$.

Elements of proof 1/3

Lemma

Assume (A), $\gamma < 0$. *Let* $\tilde{q} \in [1, \infty)$ *and*

$$0 \leq \alpha < \min\left\{\frac{1}{H\widetilde{q}}, \gamma - 1 + \frac{1}{H}\right\}.$$

For any $f \in L^{\widetilde{q}'}([0,T]; \mathcal{C}_b^\infty(\mathbb{R}^d))$ *and any* $0 \leq s < t \leq T$,

$$\left|\mathbb{E}\int_s^t f_r(X_r)\, dr\right| \lesssim \|f\|_{L^{\widetilde{q}'}_{[s,t]}\mathcal{C}^{-\alpha}} \times \left((t-s)^{\frac{1}{\widetilde{q}}-\alpha H} + (\|\varphi\|_{L^q_{[s,t]}\mathcal{C}^\gamma} + \|\varphi\|^{1+\eta}_{L^q_{[s,t]}\mathcal{C}^\gamma})(t-s)^\varepsilon\right).$$

Elements of proof 2/3

Sketch of proof of the Lemma:
We introduce a Sewing Lemma with shifting (deterministic version of [Gerencsér'23]) and control functions.

Let $\widetilde{X} = X - B$ and for $u < v \leq T$ with $u - (v-u) \geq 0$,

$$A_{u,v} := \mathbb{E} \int_u^v f_r(B_r + \mathbb{E}^{u-(v-u)} \widetilde{X}_r) \, \mathrm{d}r$$

Idea: $\mathcal{A}_t = \mathbb{E} \int_s^t f_r(X_r) \, \mathrm{d}r \approx \sum A_{u_k, u_{k+1}}$.

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(a) $|A_{u,v}| \lesssim \|f\|_{L^{\widetilde{q}'}_{[u,v]} \mathcal{C}^{-\alpha}} (v-u)^{\frac{1}{\widetilde{q}} - \alpha H}$.

(b) for $\xi = \frac{u+v}{2}$,

$$\begin{aligned} &|A_{u,v} - A_{u,\xi} - A_{\xi,v}| \\ &\lesssim (\|\varphi\|_{L^q_{[u,v]} \mathcal{C}^\gamma} + \|\varphi\|^{1+\eta}_{L^q_{[u,v]} \mathcal{C}^\gamma}) \|f\|_{L^{\widetilde{q}'}_{[u,v]} \mathcal{C}^{-\alpha}} (v-u)^{H(\gamma - 1 + \frac{1}{Hq'} + \frac{1}{H\widetilde{q}} - \alpha)}. \end{aligned}$$

(c) For any $t \in [0,T]$, the convergence in probab. of $\sum_{t_i^n \in \Pi^n} A_{t_i^n, t_{i+1}^n}$ to $\mathbb{E} \int_0^t f_r(X_r) \, \mathrm{d}r$, $\forall$ partitions of $[0,t]$ s.t. $|\Pi^n| \to 0$.

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$\Rightarrow |\mathbb{E} \int_s^t f_r(X_r) \, \mathrm{d}r| \lesssim \|f\|_{L^{\widetilde{q}'}_{[s,t]} \mathcal{C}^{-\alpha}} \big((t-s)^{\frac{1}{\widetilde{q}} - \alpha H} + C_\varphi (t-s)^{H(\gamma - 1 + \frac{1}{Hq'} + \frac{1}{H\widetilde{q}'} - \alpha)}\big)$.

Elements of proof 3/3

Proving (b) boils down to control

$$\Big|\mathbb{E}\int_u^\xi \underbrace{f_r(B_r+\mathbb{E}^{u-(v-u)}\widetilde{X}_r)-f_r(B_r+\mathbb{E}^{u-(\xi-u)}\widetilde{X}_r)}_{=:\tilde{f}_r(B_r)}\,\mathrm{d}r\Big|.$$

Elements of proof 3/3

Proving (b) boils down to control

$$\Big|\mathbb{E}\int_u^\xi \underbrace{f_r(B_r+\mathbb{E}^{u-(v-u)}\widetilde{X}_r)-f_r(B_r+\mathbb{E}^{u-(\xi-u)}\widetilde{X}_r)}_{=:\tilde{f}_r(B_r)}\,\mathrm{d}r\Big|.$$

- Use that $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)|=|g_{\sigma^2}*\tilde{f}_r\left(\mathbb{E}^{u-(\xi-u)}[B_r]\right)|$ with $\sigma^2=\mathrm{Var}(B_r\mid\mathcal{F}_{u-(\xi-u)})\gtrsim(r-u+\xi-u)^{2H}$ $\leadsto$ use LND!

Elements of proof 3/3

Proving (b) boils down to control

$$\left|\mathbb{E}\int_u^\xi \underbrace{f_r(B_r+\mathbb{E}^{u-(v-u)}\widetilde{X}_r)-f_r(B_r+\mathbb{E}^{u-(\xi-u)}\widetilde{X}_r)}_{=:\tilde{f}_r(B_r)}\,\mathrm{d}r\right|.$$

- Use that $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)| = |g_{\sigma^2} * \tilde{f}_r\left(\mathbb{E}^{u-(\xi-u)}[B_r]\right)|$ with $\sigma^2 = \mathrm{Var}(B_r \mid \mathcal{F}_{u-(\xi-u)}) \gtrsim (r-u+\xi-u)^{2H}$ ⇝ use LND!
- Thus $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)| \lesssim \|\tilde{f}_r\|_{\mathcal{C}^{-\alpha-1}}(r-u+\xi-u)^{-(\alpha+1)H}$ ⇝ use smoothing of Gaussian kernel.

Elements of proof 3/3

Proving (b) boils down to control

$$\Big|\mathbb{E}\int_u^\xi \underbrace{f_r(B_r+\mathbb{E}^{u-(v-u)}\widetilde{X}_r)-f_r(B_r+\mathbb{E}^{u-(\xi-u)}\widetilde{X}_r)}_{=:\tilde{f}_r(B_r)}\,\mathrm{d}r\Big|.$$

- Use that $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)|=|g_{\sigma^2}*\tilde{f}_r\left(\mathbb{E}^{u-(\xi-u)}[B_r]\right)|$ with $\sigma^2=\mathrm{Var}(B_r\mid\mathcal{F}_{u-(\xi-u)})\gtrsim(r-u+\xi-u)^{2H}$ $\leadsto$ use LND!
- Thus $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)|\lesssim\|\tilde{f}_r\|_{\mathcal{C}^{-\alpha-1}}(r-u+\xi-u)^{-(\alpha+1)H}$ $\leadsto$ use smoothing of Gaussian kernel.
- Now $\|\tilde{f}_r\|_{\mathcal{C}^{-\alpha-1}}\leq\|f_r\|_{\mathcal{C}^{-\alpha}}|\mathbb{E}^{u-(v-u)}\widetilde{X}_r-\mathbb{E}^{u-(\xi-u)}\widetilde{X}_r|$.

Elements of proof 3/3

Proving (b) boils down to control

$$\left|\mathbb{E}\int_u^\xi \underbrace{f_r(B_r+\mathbb{E}^{u-(v-u)}\widetilde{X}_r)-f_r(B_r+\mathbb{E}^{u-(\xi-u)}\widetilde{X}_r)}_{=:\tilde{f}_r(B_r)}\,\mathrm{d}r\right|.$$

- Use that $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)| = |g_{\sigma^2} * \tilde{f}_r\left(\mathbb{E}^{u-(\xi-u)}[B_r]\right)|$ with $\sigma^2 = \mathrm{Var}(B_r \mid \mathcal{F}_{u-(\xi-u)}) \gtrsim (r-u+\xi-u)^{2H}$ ⇝ use LND!

- Thus $|\mathbb{E}^{u-(\xi-u)}\tilde{f}_r(B_r)| \lesssim \|\tilde{f}_r\|_{\mathcal{C}^{-\alpha-1}}(r-u+\xi-u)^{-(\alpha+1)H}$ ⇝ use smoothing of Gaussian kernel.

- Now $\|\tilde{f}_r\|_{\mathcal{C}^{-\alpha-1}} \leq \|f_r\|_{\mathcal{C}^{-\alpha}}|\mathbb{E}^{u-(v-u)}\widetilde{X}_r - \mathbb{E}^{u-(\xi-u)}\widetilde{X}_r|$.

- It remains to control $|\mathbb{E}^{u-(v-u)}\widetilde{X}_r - \mathbb{E}^{u-(\xi-u)}\widetilde{X}_r|$: using *Stochastic* sewing with controls,

$$\|\widetilde{X}_r-\mathbb{E}^{u-(v-u)}\widetilde{X}_r\|_{L^\infty_\Omega} \leq C(\|\varphi\|_{L^q_{[u-,r]}\mathcal{C}^\gamma}+\|\varphi\|^{1+\eta}_{L^q_{[u-,r]}\mathcal{C}^\gamma})(r-u+v-u)^{\frac{1}{q'}+H\gamma}.$$

Elements of proof - Conclusion

Denote $\mathcal{S}_{u,v}$ the set of functions $f \in L^{\widetilde{q}'}([u,v];\mathcal{C}_b^\infty)$ s.t. $\|f\|_{L^{\widetilde{q}'}_{[u,v]}\mathcal{C}^{-\alpha}} \leq 1$.

By a density argument, it is sufficient to take the supremum over $f \in \mathcal{S}_{u,v}$, to get

$$\begin{aligned}\|\mathcal{L}(X_\cdot)\|_{L^{\tilde{q}}_{[u,v]}\mathcal{B}^{\alpha}_{1,1}} &\leq C \sup_{f\in\mathcal{S}_{u,v}} \Big| \int_u^v \langle f_s, \mathcal{L}(X_s)\rangle \,\mathrm{d}s \Big| \\ &\leq C \sup_{f\in\mathcal{S}_{u,v}} \Big| \mathbb{E} \int_u^v f_s(X_s)\,\mathrm{d}s \Big|.\end{aligned}$$

It remains to use the lemma.