Kinetic McKean-Vlasov equations

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(with **Alessio Rondelli**)

Workshop on Irregular Stochastic Analysis

Cortona, 23-27 June 2025

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- W is a d-dimensional Brownian motion
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- W is a d-dimensional Brownian motion
- μ_t denotes the **law** of (X_t, V_t)
- it is a path-dependent model:

  in particular, coefficients depend on the law of $\int_0^t V_s ds$

Applications

Some fields of application:

- **Kinetic theory** (original motivation with measure-dependent σ): Fournier and Hauray (2016), Fournier and Guillin (2017)

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- **Kinetic theory** (original motivation with measure-dependent σ): Fournier and Hauray (2016), Fournier and Guillin (2017)
- **Mathematical finance**: Guyon and Henry-Labordère (2012)
- **Neurosciences**: Delarue, Inglis, Rubenthaler and Tanré (2015)
- **Mean-field games**: Carmona, Delarue et al. (2018)
- **Random matrices**: Anderson, Guionnet and Zeitouni (2010)
- **Battery models**: Guhlke, Gajewski, Maurelli, Friz and Dreyer (2018)
- **Population dynamics**: Morale, Capasso and Oelschläger (2005)
- many others (see references in Chaintron and Diez (2021))

Some known results, I

Issoglio, Pagliarani, Russo and Trevisani (2024)

$$\begin{cases} \mathrm{d}V_t = b(t, X_t, V_t, \mu_t)\mathrm{d}t + \mathrm{d}W_t \\ \mathrm{d}X_t = V_t \mathrm{d}t \end{cases}$$

- drift in anisotropic Besov spaces
- Brownian noise, constant diffusion

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- well-posedness via martingale problem

Some known results, II

Zhang (2024), Hao, Jabir, Menozzi, Röckner and Zhang (2024), Chaudru de Raynal, Jabir and Menozzi (2025)

$$\begin{cases} \mathrm{d}V_t = b(t, X_t, V_t, \mu_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)} \\ \mathrm{d}X_t = V_t \mathrm{d}t \end{cases}$$

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- propagation of chaos from moderately interacting particle approximation

Some known results, III

Veretennikov (2024), P., Rondelli and Veretennikov (2024)

$$\begin{cases} dX_{0,t} = \int b_0(t, X_t, \cdot) d\mu_{X_t} \, \mathrm{d}t \\ dX_{1,t} = \int b_1(t, X_t, \cdot) d\mu_{X_t} \mathrm{d}t + \int \sigma(t, X_t, \cdot) d\mu_{X_t} \, dW_t \end{cases}$$

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- linear growth: $|b(t,x,y)| + |\sigma(t,x,y)| \leqslant \mathbf{c}(1 + |x| + |y|)$
- continuous in $x_0, y_0 \in \mathbb{R}^{N-d}$ (degenerate variables)
- measurable in $x_1, y_1 \in \mathbb{R}^d$ (diffusion variables) and t

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- **existence**: regularization + Krylov's estimates + compactness argument
- **uniqueness**: Girsanov if $\sigma = \sigma(t, X_t)$

Some known results, IV

Well posedness in the non-degenerate case

$$dX_t = b(t, X_t, \mu_{X_t})dt + \sigma(t, X_t, \mu_{X_t})dW_t$$

Chaudru de Raynal (2020), Chaudru de Raynal and Frikha (2022):

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Tools: Zvonkin transform, backward PDE on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, stability of heat kernels via Levi's parametrix method

Anisotropic distances

Anisotropic (capped) distance: for $0 < \alpha \leqslant 1$

$$\mathbf{d}_\alpha((x,v),(x',v')) := \left(|x-x'|^{\frac{\alpha}{3}} + |v-v'|^{\alpha}\right) \wedge 1$$

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Wasserstein distance (primal formulation)

$$W_{1,\alpha}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \iint \mathbf{d}_\alpha(z,\zeta)\,\pi(dz,d\zeta), \qquad \mu,\nu \in \mathcal{P}(\mathbb{R}^{2d})$$

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where

$$\|f\|_{\mathrm{Lip}_{\mathbf{d}_\alpha}} = \|f\|_\infty + \sup_{z\neq\zeta} \frac{|f(z)-f(z')|}{\mathbf{d}_\alpha(z,z')}, \qquad z=(x,v)$$

Assumptions

Dynamics

$$\begin{cases} \mathrm{d}V_t = b(t, X_t, V_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, V_t, \mu_t)\mathrm{d}W_t \\ \mathrm{d}X_t = V_t \mathrm{d}t \end{cases}$$

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Hypotheses

H1 **Hölder-type continuity:** b, σ bounded, measurable, and s.t.

$$|f(t,z,\mu) - f(t,z',\nu)| \lesssim \mathbf{d}_\alpha(z,z') + W_{1,\alpha}(\mu,\nu), \qquad z = (x,v)$$

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H2 $\sigma\sigma^*$ uniformly positive definite (**Hörmander condition**)

Main result: P. and Rondelli (2025)

Under Assumptions **H1-H2**, for any initial law $\boldsymbol{\eta_0} \in \mathcal{P}(\mathbb{R}^{2d})$

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Task

Circumvent PDEs involving derivatives w.r.t. the measure variable

Linearized MKV equation, I

For any *fixed* $\mu \in C\left([0,T];\mathcal{P}(\mathbb{R}^{2d})\right)$, the *linear (non-MKV)* SDE

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Goal: show that the map

$$\mu \longmapsto m^\mu \equiv \text{flow of marginals of } (X^\mu, V^\mu)$$

is a contraction on the complete metric space $C\left([0,T];\mathcal{P}(\mathbb{R}^{2d})\right)$ endowed with the distance

$$\mathbf{W}_{1,\alpha}(\mu,\nu) := \sup_{t\in[0,T]} W_{1,\alpha}(\mu_t,\nu_t)$$

Linearized MKV equation, II

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The Fokker-Plank operator of the linearized SDE is $\mathcal{A}^{\mu} + \mathbf{Y}$ with

$$\mathcal{A}_t^{\mu} := \frac{1}{2}\sum_{i,j=1}^{d} c_{ij}(t,x,v,\mu_t)\partial_{v_i v_j} + b(t,x,v,\mu_t)\cdot\nabla_v$$

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- $\mathcal{A}^{\mu} + \mathbf{Y}$ has a fundamental solution $p^{\mu}(t,x,v;T,X,V)$

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Menozzi (2018): weak well-posedness of the linearized SDE

Push-forward and pull-back operators

Push-forward

$$\vec{P}^{\mu}_{0,t}\eta = \int p^{\mu}(0,z;t,\cdot)\,\eta(\mathrm{d}z)$$

Pull-back

$$\overleftarrow{P}^{\mu}_{t,T}\eta = \int p^{\mu}(t,\cdot;T,\zeta)\,\eta(\mathrm{d}\zeta)$$

Idea

Express $\mathbf{W}_{1,\alpha}$ via push-forward and pull-back operators of $\eta \in \mathcal{P}(\mathbb{R}^{2d})$.

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Wasserstein distance between the flows of marginals

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Inversion Lemma (duality)

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$$\int f \, \mathrm{d}\big(\vec{P}^{\mu}_{0,t} - \vec{P}^{\nu}_{0,t}\big)\, \boldsymbol{\eta_0} = \int_0^t \int \Big(\overleftarrow{P}^{\mu}_{0,s} \left(\mathcal{A}^{\mu}_s - \mathcal{A}^{\nu}_s\right) \overleftarrow{P}^{\nu}_{s,t} f \Big) \, \mathrm{d}\boldsymbol{\eta_0} \, \mathrm{d}s$$

where

$$\mathcal{A}^{\mu}_t = \frac{1}{2} \sum_{i,j=1}^{d} c_{ij}(t, x, v, \mu_t) \partial_{v_i v_j} + \dots$$

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By Hölder cont. and Gaussian estimates for 2nd-order derivatives of p^{μ}

$$\mathbf{W}_{1,\alpha}\big(m^{\mu}, m^{\nu}\big) \lesssim \mathbf{W}_{1,\alpha}(\mu,\nu) \int_0^t \frac{1}{(t-s)^{1-\frac{\alpha}{2}}}\, ds \;\lesssim\; t^{\frac{\alpha}{2}} \mathbf{W}_{1,\alpha}(\mu,\nu),$$

which proves the thesis.

Thank you for your attention!

Next time: propagation of chaos (work in progress).

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