A general framework for an approximation method for invariant measures of stochastic equations

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We want to prove that in general convergence to the invariant measure + approximation scheme= simulation method with error rates (in Wasserstein distance)

Pages, Lamberton, Lemaire, Panloup (decreasing step Euler approximation to the invariant measure)

General framework : $(B, \|x\|)$ Banach space, d distance equivalent with the norm:

$$c\,\|x\| \leq d(x,0) \leq C\,\|x\|$$

Wasserstein distance

$$W_p(\mu,\nu) = (\inf_{\Pi_{\mu,\nu}} \int_{B\times B} \|x-y\|^p\,\Pi_{\mu,\nu}(dx,dy))^{1/p}$$

$$W_{d,p}(\mu,\nu) = (\inf_{\Pi_{\mu,\nu}} \int_{B\times B} d(x,y)^p \Pi_{\mu,\nu}(dx,dy))^{1/p}$$

$$\mathcal{P}_p(B) = \{\mu : \int_B \|x\|^p\,\mu(dx) < \infty\}.$$

Family of transport applications

$$\theta_{s,t} : \mathcal{P}_p(B) \to \mathcal{P}_p(B), \quad s < t$$

Example 1: $(P_t)_{t\geq 0}$ Markov semigroup, $\mu \to \theta_{s,t}(\mu) = \mu P_{t-s}$

$$\int_B f(x)\theta_{s,t}(\mu)(dx) = \int_B P_{t-s}f(x)\mu(dx)$$

Example 2 Mc-Kean Vlasov

$$X_{s,t} = X + \int_s^t f(X_{s,r}, \mathcal{L}(X_{s,r}))dr + \int_s^t \sigma(X_{s,r}, \mathcal{L}(X_{s,r}))dB_r$$

$$\mathcal{L}(X) = \mu \to \theta_{s,t}(\mu) = \mathcal{L}(X_{s,t}).$$

Example 3: Diffusion: One step Euler

$$X_{s,t} = X + \int_s^t f(X, \mathcal{L}(X))dr + \int_s^t \sigma(X, \mathcal{L}(X))dB_r$$

$$\mathcal{L}(X) = \mu \to \Theta_{s,t}(\mu) = \mathcal{L}(X_{s,t}).$$

$\pi = \{t_0 < t_1 < ... < t_n < ...\}$

$$\Theta^{\pi}_{t_0,t_n}(\mu) = \Theta_{t_{n-1},t_n} \circ ... \circ \Theta_{t_0,t_1}(\mu).$$

Definition

Let $b_* > 0$ and $\varepsilon > 0$. We say that Θ^1 and Θ^2 are (p, b_*, ε)−coupled if for some $C_* > 0$ and $h \in (0,1)$ and every $s \le t \le s+h$ and $\mu^i \in \mathcal{P}_p(B)$, $i = 1, 2$,

$$W_p^p(\Theta^1_{s,t}(\mu^1), \Theta^2_{s,t}(\mu^2)) \le (1 - b_*(t-s)) W_p^p(\mu^1, \mu^2) + C_*(1 + \|\mu_1\|_p^p + \|\mu_2\|_p^p)(t-s)^{1+\varepsilon}.$$

Last, we say that a family $\Theta_{s,t} : \mathcal{P}_p(B) \to \mathcal{P}_p(B)$, $0 \le s \le t$, is (p, b_*, ε)−self-coupled if the above holds with $\Theta^1 = \Theta^2 = \Theta$.

Hypothesis 2 (Boundedness)

$$M_p(\Theta, \mu, \delta) = \sup_{\pi : t_i - t_{i-1} \le \delta} \sup_n \left\| \Theta^\pi_{t_0, t_n}(\mu) \right\|_p^p < \infty$$

Foster Lyapunov criterion

$$\int_B \|x\|^p \, \Theta_{s,t}(\mu)(dx) \le (1 - b(t-s)) \int_B \|x\|^p \, \mu(dx) + C(t-s).$$

Approximations: The main idea

Euler scheme Idea of the proof is the semigroup argument. $a_k = W_p^p(\Theta^{1,\pi}_{t_0,t_k}(\mu^1), \Theta^{2,\pi}_{t_0,t_k}(\mu^2))$

$$a_n = e^{-b(t_n-t_0)}a_0 + \sum_{k=1}^{n}(e^{-b(t_n-t_k)}a_k - e^{-b(t_n-t_{k-1})}a_{k-1}).$$

Time grid Given $\gamma_k \downarrow 0$ and $t_k = \gamma_1 + ...\gamma_k \uparrow \infty, k \in N$.

$$\omega = \overline{\lim_n} \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2} < \infty$$

If $b \geq \omega\varepsilon$ for some $\varepsilon > 0$ then

$$\sigma_n(b, \varepsilon) = \sum_{i=1}^{n} e^{-b(t_n-t_i)}\gamma_i^{1+\varepsilon} \leq C_{b,\varepsilon} \times \gamma_n^{\varepsilon}.$$

Typically: $\gamma_n : \frac{1}{n}, \omega = 1, t_k = O(\ln k)$

Lemma

B *For the particular case* $\gamma_n = \frac{1}{n+h^{-1}}$ *one has* $\overline{\omega}(\pi) = 1$. *If* $b \neq \varepsilon$ *then* $\sigma_{b,\varepsilon}(n) \leq C_{b,\varepsilon} n^{-b\wedge\varepsilon}$ *and if* $b = \varepsilon$ *then* $\sigma_{b,\varepsilon}(n) \leq C_{b,\varepsilon} n^{-\varepsilon} \ln(1+n)$.

C *Let* $s < t$ *and* $t_n = s + \sum_{i=1}^{n} \gamma_i$. *We take* $n = n(t)$ *such that* $t_n \leq t < t_{n+1}$. *Then one may find a constant that we still denote* $C_{b,\varepsilon}$ *such that*

$$\sigma_{b,\varepsilon}(n) \leq C_{b,\varepsilon} e^{-b\wedge\varepsilon(t-s)}(1 + (t-s)^{1_{b=\varepsilon}}). \tag{1}$$

Lemma

Let Θ^1 *and* Θ^2 *be families which are* (p, b_*, ε)*-coupled for some* $b_*, \varepsilon > 0$. *Let also assume that* Θ^1 *is* $p-$*bounded and* $M_p(\Theta^1, \mu, h) < \infty$. *Then,*

$$\begin{aligned} &W_p^p(\Theta^{1,\pi}_{t_0,t_n}(\mu^1), \Theta^{2,\pi}_{t_0,t_n}(\mu^2)) \\ &\leq e^{-b_*(t_n-t_0)} W_p^p(\mu^1,\mu^2) + 2^{p+1} C_*(1 + M_p(\Theta^1,\mu^1,h) + W_p^p(\mu_1,\mu_2))\sigma_{b_*,\varepsilon}(n). \end{aligned} \tag{2}$$

Besides, $M_p(\Theta^2, \mu, h) < \infty$ *for every* $\mu \in \mathcal{P}_p(B)$ *and* Θ^2 *is* $p-$*bounded.*

With the above set up

$$e^{-b_*(t_n-t_0)} = n^{-b_*}, \quad \sigma_{b_*,\varepsilon}(n) \leq C n^{-\varepsilon}$$

Corollary

Let $\theta_{s,t}$ be a flow which is p-bounded and is (p, b_*, ε)-self-coupled for some $b_*, \varepsilon > 0$. We note $C_* > 0$, $h \in (0,1)$. Then, for$\mu^1, \mu^2 \in \mathcal{P}_p(B)$ and $s < t$

$$W_p^p(\theta_{s,t}(\mu^1), \theta_{s,t}(\mu^2)) \leq Ae^{-b_* \wedge \varepsilon(t-s)}(1 + (t-s)^{1_{b_*=\varepsilon}})$$

where $A = \Big((2^{p+1}C_*\tilde{C}_{b_*,\varepsilon} + 1)W_p^p(\mu^1, \mu^2) + 2^{p+1}C_*\tilde{C}_{b_*,\varepsilon}(1 + M_p(\theta, \mu^1))\Big)$ and $\tilde{C}_{b_*,\varepsilon} = C_{b_*,\varepsilon} + 1$.

Time homogeneous flow:

$$\begin{aligned}\theta_{r,t} \circ \theta_{s,r} =& \theta_{s,t}, s \leq r \leq t\\ \theta_{s,t}(\mu) =& \theta_{0,t-s}(\mu) \text{ for } 0 \leq s \leq t.\end{aligned}$$

Continuity in W_p: for every fixed $s \leq t$

$$\lim_{n\to\infty} W_p(\mu_n, \mu) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} W_p(\theta_{s,t}(\mu_n), \theta_{s,t}(\mu)) = 0. \tag{3}$$

Example Let us consider the following time-dependent O-U process

$$X_{s,t} = X_s + \int_s^t k(\zeta(u) - X_u)du + \sigma(W_t - W_s), \ s \leq t,$$

with $k, \sigma > 0$ and $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ bdd. Here, $X_s \in L^2$ independent of $(W_t - W_s, t \geq s)$.Then

$$X_{s,t} = X_s e^{-k(t-s)} + \int_s^t \zeta(u)e^{-k(t-u)}du + \sigma \int_s^t e^{-k(t-u)}dW_u$$

The associated flow $\theta_{s,t}(\mu) = \mathcal{L}(X_{s,t})$ when $X_s \sim \mu \in \mathcal{P}_2(\mathbb{R})$ satisfies

$$W_2^2(\theta_{s,t}(\mu_1), \theta_{s,t}(\mu_2)) \leq W_2^2(\mu_1, \mu_2)e^{-2k(t-s)}.$$

Denote by $\nu_{s,t} = \mathcal{N}(\int_s^t \zeta(u)e^{-k(t-u)}du, \frac{\sigma^2}{2k})$ the law of

$$\int_s^t \zeta(u)e^{-k(t-u)}du + \sigma \int_s^t e^{-k(t-u)}dW_u + \sqrt{\frac{\sigma^2}{2k}e^{-2k(t-s)}}Y$$

with $Y \sim \mathcal{N}(0, 1)$ independent of the other variables, and therefore

$$W_2^2(\theta_{s,t}(\mu), \nu_{s,t}) \leq 2\mathbb{E}[X_s^2]e^{-2k(t-s)} + \frac{\sigma^2}{k}e^{-2k(t-s)} =_{t\to\infty} O(e^{-2k(t-s)}).$$

However, we can choose ζ so that $\int_s^t \zeta(u)e^{-k(t-u)}du$ does not converge as $t \to \infty$ (take for example $\zeta(u) = \cos(u)$) and therefore $\nu_{s,t}$ (and thus $\theta_{s,t}(\mu)$) does not converge in law.

So from now on, we only consider continuous time homogeneous flows.

Proposition $\theta_{s,t} : \mathcal{P}_p(B) \to \mathcal{P}_p(B)$ which is p-bounded and is (p, b_*, ε) self-coupled for some $h \in (0,1)$, $b_*, \varepsilon > 0$. Then, there exists a unique $\nu \in \mathcal{P}_p(B)$ s.t. for $s \leq t$ one has $\nu = \theta_{s,t}(\nu)$. It satisfies $\|\nu\|_p^p = M_p(\theta, \nu)$ and we have for every $s \leq t$ and every $\mu \in \mathcal{P}_p(B)$,

$$W_p^p(\theta_{s,t}(\mu), \nu) \leq Ae^{-\varepsilon \wedge b_*(t-s)}(1 + (t-s))^{\mathbf{1}_{b_*=\varepsilon}}$$

Theorem Let θ be p-bounded and is (p, b_*, ε) self-coupled for some $h \in (0,1)$, $b_*, \varepsilon > 0$ and ν the invariant measure. Let $\Theta_{s,t} : \mathcal{P}_p(B) \to \mathcal{P}_p(B)$, $s \leq t$, be a family of applications which is (p, b_*, ε) coupled with $\theta_{s,t}$. Let $\gamma_n = \frac{1}{n+h^{-1}}$ and $t_n = s + \gamma_1 + ... + \gamma_n$, $n \in \mathbb{N}$. Then, we have for every $\mu \in \mathcal{P}_p(B)$

$$W_p^p(\Theta^{\pi}_{s,t_n}(\mu), \nu) \leq An^{-b_* \wedge \varepsilon}(1 + \ln(n+1)^{\mathbf{1}_{b_*=\varepsilon}})$$

with $A = 2^{2p}C_*(1 + M_p(\theta, \mu))(2C_{b_*,\varepsilon} + 1)$.

JOINT probabilistic representation

$$W_p^p(\mu^1, \mu^2) = \mathbb{E}(d^p(X^1, X^2)),$$

$$\mathbb{E}(d^p(\mathcal{X}^1_{s,t}, \mathcal{X}^2_{s,t})) \leq (1 - b_*(t-s))\mathbb{E}(d^p(X^1, X^2)) + C_*(1 + \|X^1\|_p^p + \|X^2\|_p^p)(t-s)^{1+\varepsilon}$$

Lemma

Let $\Theta^i_{s,t} : \mathcal{P}_1(B) \to \mathcal{P}_1(B)$, $s \le t$, $i = 1, 2$ *be two families of applications such that:*

- Θ^1 *is* 1*-bounded and* $(1, b_*, \varepsilon)$ *self-coupled,*
- *There exists* $C \in \mathbb{R}_+$ *such that for all* $s \le t$ *and* $\mu \in \mathcal{P}_1(B)$, $W_1(\Theta^1_{s,t}(\mu), \Theta^2_{s,t}(\mu)) \le C(1 + \|\mu\|_1)(t - s)^{1+\varepsilon}$.

Then, Θ^1 *and* Θ^2 *are* $(1, b_*, \varepsilon)$*-coupled. Furtherore,* Θ^2 *is* 1*-bounded and* $(1, b_*, \varepsilon)$ *self-coupled as well.*

Example 1 Langevin equation (Eberle)

$$X_{s,t} = X + \int_s^t b(X_{s,r})dr + \int_s^t \sigma dB_r$$

We assume that

$$A_1 \quad |b(x) - b(y)| \leq L_b |x - y|$$

$$A_2 \quad \langle x - y, b(x) - b(y) \rangle \leq -\kappa |x - y| \quad |x - y| \geq R$$

Eberle constructs **explicit function** $f : R_+ \to R_+$ (depending on κ, L_b, R) and defines the distance

$$d_f(x, y) = f(|x - y|).$$

Proposition (Eberle) One may find b_* such that

$$W_{d_f,1}(\theta_{s,t}(\mu), \theta_{s,t}(\nu)) \leq e^{-b_*(t-s)} W_{d_f,1}(\mu, \nu).$$

Remark 1 d_f is equivalent with the Euclidean norm $|\cdot|$. We get easily

$$W_1(\theta_{s,t}(\mu), \theta_{s,t}(\nu)) \leq \frac{1}{c_*} e^{-b_*(t-s)} W_{d_f,1}(\mu, \nu)$$

but then the multiplicative constant becomes $1/c_*$ which is in general strictly greater that one

Corollary

$$W_{d_f,1}(\Theta^{\pi}_{t_0,t_n}(\mu),\nu) \leq Cn^{-b_*\wedge\frac{1}{2}}.$$

The factor 1/2 is due to the approximation rate.

Proof Define the Euler scheme

$$\overline{X}_{s,t} = X + \int_s^t b(X)dr + \int_s^t \sigma dB_r \quad \overline{P}_{s,t}(\mathcal{L}(X)) = \mathcal{L}(\overline{X})$$

One checks

$$W_{d_f,1}(\theta_{s,t}(\mu),\Theta_{s,t}(\mu)) \leq CW_1(\theta_{s,t}(\mu),\Theta_{s,t}(\mu)) \leq CE\left|X_{s,t} - \overline{X}_{s,t}\right| \leq C\,|t-s|^{3/2}$$

so that applying previous Lemma

$$W_1(\Theta(\mu)^{\pi}_{t_0,t_n},\nu) \leq CW_{d_f,1}(\Theta(\mu)^{\pi}_{t_0,t_n},\nu) \leq Cn^{-b_*\wedge\frac{1}{2}}.$$

A similar result holds for the degenerate system (Schuch)

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= (ub^E(X_t) + u\int_{\mathbb{R}^d} b^I(X_t, z)\mu_t^X(dz) - \gamma Y_t)dt + \sqrt{2\gamma u}dB_t \end{aligned}$$

Under the conditions $g : \mathbb{R}^d \to \mathbb{R}^d$ with Lipschitz constant L_g such that $b^E(x) = -Kx + g(x)$. Moreover, the function g verifies for some $R > 0$,

$$\langle x_1 - x_2, g(x_1) - g(x_2)\rangle \leq 0 \quad \text{if} \quad |x_1 - x_2| \geq R.$$

The above assumption implies in particular that

$$\left\langle x_1 - x_2, b^E(x_1) - b^E(x_2)\right\rangle \leq -\kappa\, |x_1 - x_2|^2 \quad \text{if} \quad |x_1 - x_2| \geq R.$$

Bolzmann type equations. Stochastic equation

We consider the stochastic equation: (Recall v :speed, z: angle,)

$$X_{s,t}(X) = X + \int_s^t b(X_{s,r}, \mathcal{L}(X_{s,r}))dr \tag{4}$$

$$+ \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X_{s,r-}, \mathcal{L}(X_{s,r}))1_{\{u \le \gamma(v,z,X_{s,r-})\}} N_{\mathcal{L}(X_{s,r})}(dv, dz, du, dr). \tag{5}$$

With $N_{\mathcal{L}(X_{s,r})}$ a Poisson point measure with compensator

$$\widehat{N}_{\mathcal{L}(X_{s,r})}(dv, dz, du, dr) = \mathcal{L}(X_{s,r})(dv)\nu(dz)dudr$$

We define

$$\mathcal{L}(X) = \mu \quad \rightarrow \quad \theta_{s,t}(\mu) = \mathcal{L}(X_{s,r}(X))$$

One step Euler scheme: Given $\rho \in \mathcal{P}_1$ we take $X \sim \rho$ and construct

$$Y_{s,t}(X) = X + b(X,\rho)(t-s) + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v,z,X,\rho) 1_{\{u \leq \gamma(v,z,X)\}} N_\rho(dv,dz,du,dr)$$

Then define $\Theta_{s,t} : \mathcal{P}_1 \to \mathcal{P}_1$ by

$$\rho \to \Theta_{s,t}(\rho) = \mathcal{L}(Y_{s,t}(X)).$$

And we have (Alfonsi,Bally)

$$W_1(\theta_{s,t}(\mu), \Theta_{s,t}(\mu)) \leq C(\mu)(t-s)^2.$$

Hypothesis

A1

$$|b(x,\mu) - b(y,\nu)| \leq C_b(|x-y| + W_1(\mu,\nu))$$

$$\langle x-y, b(x,\mu) - b(y,\mu)\rangle \leq -\bar{b}\,|x-y| \quad x, y \in R^d$$

A2

$$\Big|(c(v,z,x) - c(v',z,x'))\gamma(v,z,x)\Big| + \Big|(\gamma(v,z,x) - \gamma(v',z,x'))c(v,z,x)\Big|$$

$$\leq q(z)(\big|x-x'\big| + \big|v-v'\big|) \quad with \quad Q := \int q(z)d\nu(z)$$

$$|c\gamma(v,z,x)| \leq \overline{c}(z)(1+|v|+|x|).$$

A3

$$b_* := \bar{b} - 2Q > 0.$$

Theorem Assume the above conditions . Then, θ and Θ satisfy the 1-Foster-Lyapunov condition, are $(1, b_*, 1)$ coupled for $b_* = \bar{b} - 2Q$, and θ is self-coupled.

Furthermore, the Markov process $(X^1_{s,t}(X))_{t \geq s}$ admits a unique invariant measure $\nu \in \mathcal{P}_1(\mathbb{R}^d)$, and there exists $C \in \mathbb{R}_+$ such that for every $\mu \in \mathcal{P}_1(\mathbb{R}^d)$, $n \geq 1$ and $t > s$,

$$W_1(\Theta^\pi_{s,t_n}(\mu), \nu) \leq C(\|\mu\|_1 + 1)\frac{(1 + \ln(n+1)^{\mathbf{1}_{b_*=1}})}{n^{b_* \wedge 1}},$$
$$W_1(\theta_{s,t_n}(\mu), \nu) \leq C(\|\mu\|_1 + 1) \times (1 + (t_n - s)^{\mathbf{1}_{b_*=1}})e^{-b_* \wedge 1(t-s)},$$

where $t_k = s + \sum_{i=1}^k \frac{1}{1+i}$ and $\pi = \{t_0 < \cdots < t_n\}$.

Proof We have to check that Θ verify **H 2 (Boundedness)** and **H 1 ((b, ε)−self-coupling)**

$$W_1(\Theta_{s,t}(\mu^1), \Theta_{s,t}(\mu^2)) \leq (1 - b(t-s))W_1(\mu^1, \mu^2) + C(t-s)^2$$

Step 1 Coupling

a) Π optimal coupling of μ^1, μ^2 and $\overline{X} = (\overline{X}^1, \overline{X}^2)$ of law Π so that

$$E\left|\overline{X}^1 - \overline{X}^2\right| = W_1(\mu^1, \mu^2)$$

b) $\tau = (\tau^1, \tau^2) : (0,1) \to R^d \times R^d$ such that

$$\int_0^1 \Phi(\tau(w))dw = E(\Phi(\overline{X}^1, \overline{X}^2))$$

Step 2 Objective Poisson Point measure $N(dw, du, dz, dr)$ with

$$\widehat{N}(dw, du, dz, dr) = dw \times du \times \nu(dz) \times dr.$$

Equations (On the same probablity space): for $i = 1, 2$

$$\overline{X}^i_{s,t}(\mu^i) = \overline{X}^i + b(\overline{X}^i, \mu^i)(t - s) + \int_s^t \int_{(0,1)\times E\times\mathbb{R}_+} c(\tau^i(w), z, \overline{X}^i, \mu^i) 1_{\{u \le \gamma(v,z,\overline{X}^i)\}} N(dw, dz, du, dr)$$

Then

$$\mathcal{L}(\overline{X}^i_{s,t}) = \Theta_{s,t}(\mu^i)$$

so that

$$W_1(\Theta_{s,t}(\mu^1), \Theta_{s,t}(\mu^2)) \le E\left|\overline{X}^1_{s,t} - \overline{X}^2_{s,t}\right|.$$

Remark We are on the same probability space so we may use an L^1 calculus.

Step 3 Itô

$$\begin{aligned} W_1(\Theta_{s,t}(\mu^1), \Theta_{s,t}(\mu^2)) &\le E\left|\overline{X}^1_{s,t} - \overline{X}^2_{s,t}\right| \\ &\le (1 - b_*(t-s))E\left|\overline{X}^1 - \overline{X}^2\right| + C(t-s)^2 \\ &= (1 - b_*(t-s))W_1(\mu^1, \mu^2) + C(t-s)^2. \end{aligned}$$

$y_{s,t} = \bar{X}^1_{s,t} - \bar{X}^2_{s,t}$. Using Itô's formula (again, one has to take first a regularization and then to pass to the limit) we get

$$\begin{aligned} \mathbb{E}\left|y_{s,t}\right| &= \mathbb{E}\left|X^1 - X^2\right| + \mathbb{E}\int_s^t \left\langle \frac{y_{s,r}}{\left|y_{s,t}\right|}, b(X_1) - b(X_2)\right\rangle dr \\ &+\mathbb{E}\int_s^t \int_E \int_{\mathbb{R}_+} \int_0^1 \left|y_{s,r-} + \Delta q(w,u,z)\right| - \left|y_{s,r-}\right| dN(w,u,z,r) \quad (6) \end{aligned}$$

with $\Delta q(w,u,z) = q(\tau^1(w), u, z, X^1) - q(\tau^2(w), u, z, X^2)$.

Neuronal model in W_1

mean field type stochastic equation

$$X_t = X_0 + \int_0^t b(X_s)ds + J\int_0^t \mathbb{E}[f(X_s)]ds - \int_0^t \int_{\mathbb{R}_+} X_{u-} 1_{z \leq f(X_{u-})\wedge M} dN(u,z),$$

where N is a random Poisson measure[1] with compensator measure given by the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$. Furthermore, $M > 0$ and $b, f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy:

$$\forall x, y \geq 0, |b(x) - b(y)| \leq C_b|x-y|,\ |f(x) - f(y)| \leq C_f|x-y| \text{ and } f(0) = 0.$$

Contraction:

there exists $\bar{b} > 0$ such that for any $x, y \geq 0$,

$$\begin{aligned} &\operatorname{sgn}(y-x)(b(y) - b(x)) - |y-x|(f_M(y) \vee f_M(x)) \\ &+ x(f_M(y) - f_M(x))^+ + y(f_M(x) - f_M(y))^+ \leq -\bar{b}\,|x-y| \end{aligned}$$

Note that this is possible even with $b \equiv 0$.

Assume $\bar{b} - C_f J > 0$. Then, θ and Θ satisfy the 1-Foster-Lyapunov condition, are $(1, b_*, 1)$ coupled for every $b_* = \bar{b} - C_f J$, and θ is self-coupled.

Furthermore, the Markov process $(X_{s,t}(X))_{t \geq s}$ admits a unique invariant measure $\nu \in \mathcal{P}_1(\mathbb{R}_+)$, and there exists $C \in \mathbb{R}_+$ such that for every $\mu \in \mathcal{P}_1(\mathbb{R}_+)$, $n \geq 1$ and $t > s$,

$$W_1(\Theta^{\pi}_{s,t_n}(\mu), \nu) \leq C(\|\mu\|_1 + 1)\frac{(1 + \ln(n+1))^{\mathbf{1}_{b_*=1}}}{n^{b_* \wedge 1}},$$
$$W_1(\theta_{s,t_n}(\mu), \nu) \leq C(\|\mu\|_1 + 1) \times (1 + (t_n - s)^{\mathbf{1}_{b_*=1}})e^{-(b_* \wedge 1)(t-s)},$$

where $t_k = s + \sum_{i=1}^{k} \frac{1}{1+i}$ and $\pi = \{t_0 < \cdots < t_n\}$.

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