The nonlinear Schrödinger equation with multiplicative noise and arbitrary power of the nonlinearity

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Outline

1. The nonlinear Schrödinger equation
2. Known results
   - global existence
   - local existence (blow-up)
3. The stochastic NLS with a nonlinear multiplicative noise
4. No blow-up by noise
   - global existence of a solution
   - global existence + uniqueness of solution
5. Invariant measures
   - existence
   - uniqueness
6. Conclusion

The nonlinear Schrödinger (NLS) equation

$$\partial_t u + \mathrm{i}\Delta u + \mathrm{i}\alpha|u|^{2\sigma}u = f$$

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$\mathrm{i} = \sqrt{-1}$
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initial condition $u(0,x) = u^0(x)$

We work on $\mathbb{R}^d$ or on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$

The existence of solutions is obtained for initial data in spaces H^s for suitable s, d, α and σ.

A classical result is for initial data in H^1, based on the conservation of

mass $\mathcal{M}(u) = \|u\|_{L^2}^2$

and

energy $\mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\alpha}{2+2\sigma}\|u\|_{L^{2+2\sigma}}^{2+2\sigma}$

Actually, if there is a "good" local existence result, then the global H^1-existence of solutions follows from a priori estimates.

Assumptions to prove the global existence for $u^0 \in H^1$

- If $\alpha = 1$ (focusing): $0 \leq \sigma < \frac{2}{d}$
- If $\alpha = -1$ (defocusing): $\begin{cases} 0 \leq \sigma < \frac{2}{d-2}, & \text{for } d \geq 3 \\ \sigma \geq 0, & \text{for } d \leq 2 \end{cases}$

One uses the the continuous embeddings

$$\begin{aligned} H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2) & \qquad \forall\, p \in [2, \infty) \\ H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) & \qquad \forall\, p \in [2, \tfrac{2d}{d-2}] \text{ for } d \geq 3 \end{aligned}$$

Hence for σ chosen above, there is the continuous embedding

$$H^1(\mathbb{R}^d) \subset L^{2+2\sigma}(\mathbb{R}^d). \tag{1}$$

Working in the energy space H^1, let us denote by σ_{cr} the upper bound on the power term

$$\sigma_{cr} = \begin{cases} \frac{2}{(d-2)_+}, & \alpha = -1 \\ \frac{2}{d}, & \alpha = 1 \end{cases}$$

The global existence has been proven for $\sigma < \sigma_{cr}$
in the deterministic and in stochastic setting
in the full space $\mathbb{R}^d$ or in bounded domains (torus, ...)

Less or no results are known for $\sigma = \sigma_{cr}$ (the critical exponent) or $\sigma > \sigma_{cr}$ (the supercritical exponent).

blow-up (local but not global existence)

When working in the energy space H^1, there is blow-up for the NLS equation in the supercritical case.
For instance, there is blow-up in finite time

- in the **focusing supercritical** case: for $\frac{2}{d} \leq \sigma \leq \frac{2}{(d-2)_+}$ and $\alpha = 1$, provided the initial data $u^0 \in H^1$, $\int_{\mathbb{R}^d} |x|^2 |u^0(x)|^2 \mathrm{d}x < +\infty$ and has negative energy (see Cazenave)
- in the **defocusing supercritical** case: for $d \geq 5$ there are examples of energy supercritical parameters σ for which there is blow-up in finite time if the initial data C^∞ are well localized spherically symmetric functions (see Merle+Raphaël+Rodnianski+Szeftel, 2022)

Blow-up results have been proven for the **stochastic** NLS eq as well.

Working on the full space $\mathbb{R}^d$, De Bouard and Debussche (2002) proved that, the **supercritical focusing** NLS eq with an additive, nondegenerate and coloured-in-space noise
if

$$\frac{2}{d} \leq \sigma < \frac{2}{(d-s)_+}$$

the initial data $u^0 \in H^1$ and $\int_{\mathbb{R}^d} |x|^2 |u^0(x)|^2 \mathrm{d}x < +\infty$,
then for any time $t > 0$, either

$$\mathbb{P}(\tau^*(u^0) < t) > 0$$

or

$$\mathbb{E} \int_0^t \left(\|u(s;u^0)\|_{H^1}^2 + \|u(s;u^0)\|_{L^{2\sigma+2}}^{4\sigma+2} \right) \mathrm{d}s = +\infty.$$

Still in the focusing case, the same authors (De Bouard and Debussche, 2005) proved blow-up with a conservative, i.e. Stratonovich, noise:

$$\mathrm{i}\mathrm{d}u - (\Delta u + |u|^{2\sigma} u)\mathrm{d}t = u \circ \mathrm{d}W$$

They consider $\sigma \geq \frac{2}{d}$ (with other conditions ...) and suitable initial data.

Cauchy problem for initial data in H^s, $s > 1$

Take $s > \frac{d}{2}$. Then

the proof of the existence of a local solution is not an issue in H^s since, via the Sobolev embedding $H^s \subset L^\infty$, one easily controls the nonlinear term (with an arbitrary large power).

A challenging open problem is the existence on any time interval. Indeed, the existence of a *global* solution is harder, since a priori control in the H^1-norm of the solution no longer implies a priori control of the H^s-norm, when $d \geq 2$.
Hence, one cannot use the conservation of the mass and the energy to deduce the existence of a global solution.

There is a paper by Sy (2021, J. Math. Pures Appl.) about **global** existence for the deterministic unforced supercritical NLS eq for μ-a.e. $u_0 \in H^s(\mathbb{T}^3)$ when $s \geq 2$ and $\sigma \in \mathbb{N}$ (where μ is an invariant measure of the deterministic NLS eq).

We introduce a suitable noise term to prove the existence of a **global** solution for **any** initial data in $H^s(\mathbb{T}^d)$ ($s > \frac{d}{2}$) and any $\sigma \in \mathbb{N}$.
The noise is multiplicative and nonlinear, related to the power σ.
So, our setting is very different with respect to that of the many papers dealing with the stochastic NLS equation with additive noise or linear (multiplicative) noise.

We consider the NLS equation with stochastic forcing term

$$\begin{cases} \mathrm{d}u(t,x) + \left[\mathrm{i}\Delta u(t,x) + \mathrm{i}\alpha |u(t,x)|^{2\sigma} u(t,x)\right] \mathrm{d}t = \phi(u(t,x))\,\mathrm{d}W(t) \\ u(0,x) = u^0(x) \end{cases} \tag{2}$$

Here $W = (W(t) : t \geq 0)$ is a classical real-valued Wiener process. We study a more general case when $\alpha \in \mathbb{C}$; therefore one can view equation (2) as the stochastic version of the equation considered by Kato.

We find sufficient conditions on the diffusion coefficient ϕ for the **global-in-time** existence of the solutions to the stochastic NLS equation (2), differently from the corresponding deterministic problem (i.e. equation (2) with $\phi = 0$):

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for every $\sigma \in \mathbb{N}$, $\quad d \geq 1$, $\alpha \in \mathbb{C}$, $s > \frac{d}{2}$ and $u^0 \in H^s(\mathbb{T}^d)$,
we find a sufficient condition on ϕ, such that the NLS equation (2) has a global solution $u \in C([0, +\infty); H^s)$ (a.s.).

In particular, the global existence holds in both focusing and defocusing cases.

Our technique to prove the existence of a global solution relies on some ideas already used for ODE's and PDE's in fluid dynamics (see, e.g., Tang+Wang 2022; Bagnara+Maurelli+Xu 2025 EJP), and is based on a tightness argument for the sequence of finite-dimensional Galerkin approximations. It requires to choose a suitable Lyapunov function.

Roughy speaking, a superlinear noise coefficient "kills the growth" of the nonlinear term so to get good a priori estimates by means of a suitable Lyapunov function. This proves the non-explosion in finite-time.

The nonlinear operator

We write the nonlinearity as

$$F(u) := |u|^{2\sigma} u$$

and assume that

$$\sigma \in \mathbb{N}.$$

Lemma

(i) *Let $s \geq 0$. Then F maps the space $H^s \cap L^\infty$ into $H^s \cap L^\infty$ and*

$$\|F(u)\|_s \leq \mathbf{K} \|u\|_{L^\infty}^{2\sigma} \|u\|_s, \qquad u \in H^s \cap L^\infty. \tag{3}$$

(ii) *Let $s > \frac{d}{2}$. Then F maps H^s into H^s and, for any $u, v \in H^s$, it holds*

$$\|F(u) - F(v)\|_s \leq L \left(\|u\|_s^{2\sigma} + \|v\|_s^{2\sigma} \right) \|u - v\|_s. \tag{4}$$

The stochastic forcing term $\phi(u(t,x))\,\mathrm{d}W(t)$

H1 W is a real-valued one-dimensional Brownian motion.

H2 The diffusion coefficient ϕ is such that

- i) $\phi : H^s \to H^s$ is bounded on balls,
- ii) $\phi : H^{s'} \to H^{-s-1}$ is continuous and bounded on balls, for $\frac{d}{2} < s' < s$;

H3 The projected coefficient $P_n\phi : H_n \to H_n$ is locally Lipschitz continuous for any $n \in \mathbb{N}$.

H4 There exists a measurable function $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which is locally bounded and such that

$$\|\phi(u) - \phi(v)\|_s \le \psi(\|u\|_s, \|v\|_s)\|u - v\|_s \qquad \forall\ u, v \in H^s;$$

H5 There exist $r > 1$ and $B \in \mathbb{R}$ such that

$$|\alpha|K\|u\|^{2\sigma}_{L^\infty} + \frac{1}{2}\frac{\|\phi(u)\|^2_s}{\|u\|^2_s} \le \frac{[\Re(u, \phi(u))_s]^2}{\|u\|^4_s} + B \qquad \forall\ u \in \mathbb{B}^c_{r,H^s}$$

where K is the constant appearing in (3).

Examples of noise forcing term

We introduce the function $h : \mathbb{R}^+ \to \mathbb{C}$ as

$$h(x) = a(1+x)^{\tilde{a}} + \mathrm{i}b(1+x)^{\tilde{b}},$$

with $a \neq 0$, $b \in \mathbb{R}$ and $\tilde{a}, \tilde{b} > 1$.
Let now consider a diffusion term ϕ of the following form

$$\phi(u) = h(\|u\|_{L^\infty})u. \tag{5}$$

We then choose the parameters a, b, c, d as follows.

- When $\tilde{a} = \tilde{b} = \sigma \geq 1$, we require a and b to fulfil

$$2|\alpha|K + b^2 \leq a^2.$$

- When $\tilde{a} = \tilde{b} > \sigma \geq 1$, we require $a \neq 0$ and

$$b^2 \leq a^2$$

- Any $\tilde{a} > \max(\tilde{b}, \sigma)$ works well with $a \neq 0$.
- ...

One could consider a slightly more general example for the noise: $\phi(u) = f(u)u$ with $f : H^s \to \mathbb{C}$ given by

$$f(u) = a(1 + \|u\|_{X_1})^{\tilde{a}} + \mathrm{i}b(1 + \|u\|_{X_2})^{\tilde{b}}, \qquad (6)$$

with $a \neq 0$, $b \in \mathbb{R}$ and $\tilde{a}, \tilde{b} > 1$.
The spaces X_1 and X_2 are chosen in such a way that for some constants K_0, K_1 and K_2

$$\|u\|_{X_2} \leq K_2 \|u\|_{X_1} \leq K_1 \|u\|_{s'} \qquad \forall u \in H^s$$
$$\|u\|_{L^\infty} \leq K_0 \|u\|_{X_1} \qquad \forall u \in H^s,$$

where, as usual, we consider $\frac{d}{2} < s' < s$.
To show that conditions H1-H5 are satisfied one imposes conditions on the parameters $a, b, \tilde{a}, \tilde{b}$ that involve also the constants K_0, K_1 and K_2.

We summarize our main results as follows.

Theorem (global existence and uniqueness in H^s)

Assume H1-H5 and

$$\sigma \in \mathbb{N}, \qquad s > \frac{d}{2}. \tag{7}$$

Then, for any initial datum $u^0 \in H^s$ there exists a unique global-in-time strong solution to (8) with $\mathbb{P}$-a.s. paths in $C([0,\infty); H^s)$.

Existence and uniqueness of global strong solutions

We rewrite equation (2) in the following abstract form

$$\begin{cases} \mathrm{d}u(t) + \mathrm{i}\,[-Au(t) + \alpha F(u(t))]\,\mathrm{d}t = \phi(u(t))\,\mathrm{d}W(t), & t > 0 \\ u(0) = u^0. \end{cases} \tag{8}$$

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We introduce the finite-dimensional Galerkin approximation of the NLS equation

$$\begin{cases} du_n(t) + \mathrm{i}\,[-Au_n(t) + \alpha P_n F(u_n(t))]\,\mathrm{d}t = P_n\phi(u_n(t))\,\mathrm{d}W(t) \\ u_n(0) = P_n(u^0) \end{cases} \tag{9}$$

where $P_n : H \to H_n = Span\{e_k : |k| \leq n\}$.

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These bounds allow to infer the tightness of the laws defined by the solution of the Galerkin approximation.
Then we prove the convergence of the Galerkin approximations to the martingale solution of the NLS equation.
Finally we prove pathwise uniqueness of the solution, from which we also infer that the solution is strong in the probabilistic sense.

The Galerkin approximation

Proposition

*Assume H1-H5, $\sigma \in \mathbb{N}$ and $s > \frac{d}{2}$.
Then for any $n \in \mathbb{N}$ and $u^0 \in H^s$ there exists a unique solution u_n of (9) defined on the time interval $[0, +\infty)$ and with $\mathbb{P}$-a.e. path in $C([0, +\infty); H_n)$. Moreover, for any T, $0 < \beta < \frac{1}{2}$ and $\delta > 0$, there exists $C > 0$ such that*

$$\sup_{n \in \mathbb{N}} \mathbb{P}\left(\|u_n\|_{L^\infty(0,T;H^s)} \geq C\right) \leq \delta. \tag{10}$$

and

$$\sup_{n \in \mathbb{N}} \mathbb{P}\left(\|u_n\|_{C^{0,\beta}([0,T];H^{-s-1})} \geq C\right) \leq \delta. \tag{11}$$

We prove that there exists a solution, globally defined in time. Local existence, i.e. existence on $[0, \tau_n)$, is a classical result. The global one is obtained by means of the **Khasmiskii's test for non explosion**.

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The idea is as follows. We introduce a sequence $\{\tau_{n,k}\}_{k\in\mathbb{N}}$ of stopping times defined by

$$\tau_{n,k} := \inf\{t \geq 0 : \|u_n(t)\|_s \geq k\}, \qquad k \in \mathbb{N}.$$

In order to prove that $\tau_n = +\infty$, $\mathbb{P}$-a.s., it is sufficient to find a Lyapunov C^2-function $\mathscr{V} : H^s \to \mathbb{R}$ satisfying

$$\begin{aligned} &\mathscr{V} \geq 0 \qquad \text{on} \quad H^s, \\ &a_k := \inf\{\mathscr{V}(u) : \|u\|_s \geq k\} \to \infty, \qquad \text{as} \quad k \to \infty, \\ &\mathscr{V}(u_n(0)) < \infty, \end{aligned}$$

such that

$$\mathbb{E}[\mathscr{V}(u_n(t \wedge \tau_{n,k}))] \leq \mathscr{V}(u_n(0)) + Ct, \tag{12}$$

for a constant $C < \infty$ and all $t \in [0, T]$ and $k \in \mathbb{N}$.

Then

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DETAILS

$$\begin{aligned}\mathbb{P}(\tau_{n,k} < t) = \mathbb{E}\left(1_{\{\tau_{n,k}<t\}}\right) &\leq \frac{1}{a_k}\mathbb{E}\left[1_{\{\tau_{n,k}<t\}}\mathscr{V}(u_n(t \wedge \tau_{n,k}))\right] \\ &\leq \frac{\mathscr{V}(u_n(0)) + Ct}{a_k}\end{aligned}$$

Passing to the limit as $k \to +\infty$, we get

$$\lim_{k\to\infty} \mathbb{P}(\tau_{n,k} < t) = 0,$$

for every fixed $t \geq 0$. Therefore $\mathbb{P}(\tau_n < t) = \lim_{k\to\infty} \mathbb{P}(\tau_{n,k} < t) = 0$ for every fixed $t \geq 0$, which means $\mathbb{P}(\tau_n = +\infty) = 1$.

As Lyapunov function we consider

$$\mathscr{V}(u) = I(\|u\|_s)$$

where $I : [0, +\infty) \to [a, +\infty)$ is a non-decreasing C^2-function such that

$$\begin{cases} I(\rho) = a, & 0 \le \rho < R \\ I(\rho) = \log_e \rho, & \rho > 2R \end{cases} \tag{13}$$

Here $R > \frac{1}{2}$ and $a \in (0, \log(2R))$.

As Lyapunov function we consider

$$\mathscr{V}(u) = l(\|u\|_s)$$

where $l : [0, +\infty) \to [a, +\infty)$ is a non-decreasing C^2-function such that

$$\begin{cases} l(\rho) = a, & 0 \leq \rho < R \\ l(\rho) = \log_e \rho, & \rho > 2R \end{cases} \tag{13}$$

Here $R > \frac{1}{2}$ and $a \in (0, \log(2R))$.

In order to get

$$\mathbb{E}[\mathscr{V}(u_n(t \wedge \tau_{n,k}))] \leq \mathscr{V}(u_n(0)) + Ct$$

we apply the Itô formula to $\mathscr{V}(u_n)$, up to the maximal existence time of the process u_n.

We obtain

$$\mathrm{d}\mathscr{V}(u_n(t)) = (\mathscr{L}_n\mathscr{V})(u_n(t))\,\mathrm{d}t + \mathscr{V}'(u_n(t))[P_n\phi(u_n(t))]\,\mathrm{d}W(t), \tag{14}$$

where $(\mathscr{L}_n\mathscr{V})(u_n(t))$ and $\mathscr{V}'(u_n(t))$ vanish when $\|u_n(t)\|_s < R$. However, when $\|u_n(t)\|_s \geq R$ they are given by

$$(\mathscr{L}_n\mathscr{V})(u_n) = \mathscr{V}'(u_n)[\mathrm{i}Au_n - \mathrm{i}\alpha P_n(F(u_n))] + \frac{1}{2}\mathscr{V}''(u_n)[P_n\phi(u_n), P_n\phi(u_n)],$$

where

$$\mathscr{V}'(u_n)[h] = l'(\|u_n\|_s)\frac{\Re\left(u_n, h\right)_s}{\|u_n\|_s}$$

and

$$\mathscr{V}''(u_n)[h,k] = l''(\|u_n\|_s)\frac{\Re\left(u_n, h\right)_s\Re\left(u_n, k\right)_s}{\|u_n\|_s^2} + l'(\|u_n\|_s)\left(\frac{\Re\left(h, k\right)_s}{\|u_n\|_s} - \frac{\Re\left(u_n, h\right)_s\Re\left(u_n, k\right)_s}{\|u_n\|_s^3}\right).$$

We notice the following simplification:

$$\begin{aligned}\mathscr{V}'(u_n)[\mathrm{i}Au_n - \mathrm{i}\alpha P_n(F(u_n))] &= \frac{I'(\|u_n\|_s)}{\|u_n\|_s}\Re\big(u_n, \mathrm{i}Au_n - \mathrm{i}\alpha P_n(F(u_n))\big)_s \\ &= \frac{I'(\|u_n\|_s)}{\|u_n\|_s}\Re\big(u_n, -\mathrm{i}\alpha P_n(F(u_n))\big)_s \\ &= \frac{I'(\|u_n\|_s)}{\|u_n\|_s}\Im\big(u_n, \alpha P_n(F(u_n))\big)_s,\end{aligned}$$

since the operators A and $(I+A)^{\frac{s}{2}}$ commute and so

$$\Re\big(u_n, \mathrm{i}Au_n\big)_s = 0.$$

Hence

$$
\begin{aligned}
&(\mathscr{L}_n\mathscr{V})(u_n)\\
&= \frac{l'(\|u_n\|_s)}{\|u_n\|_s}\Im\big(u_n, \alpha P_n(F(u_n))\big)_s + \frac{1}{2}l''(\|u_n\|_s)\frac{[\Re\big(u_n, P_n\phi(u_n)\big)_s]^2}{\|u_n\|_s^2}\\
&\qquad + \frac{1}{2}l'(\|u_n\|_s)\left(\frac{\|P_n\phi(u_n)\|_s^2}{\|u_n\|_s} - \frac{[\Re\big(u_n, P_n\phi(u_n)\big)_s]^2}{\|u_n\|_s^3}\right)\\
&\leq |\alpha|Kl'(\|u_n\|_s)\|u_n\|_{L^\infty}^{2\sigma}\|u_n\|_s + \frac{1}{2}l''(\|u_n\|_s)\frac{[\Re\big(u_n, P_n\phi(u_n)\big)_s]^2}{\|u_n\|_s^2}\\
&\qquad + \frac{1}{2}l'(\|u_n\|_s)\left(\frac{\|P_n\phi(u_n)\|_s^2}{\|u_n\|_s} - \frac{[\Re\big(u_n, P_n\phi(u_n)\big)_s]^2}{\|u_n\|_s^3}\right)
\end{aligned}
\tag{15}
$$

where K is the constant in the estimate (3).

We will now show that

$$\sup_{n\in\mathbb{N}} \sup_{u\in H_n} (\mathscr{L}_n\mathscr{V})(u_n) < \infty. \quad (16)$$

Since $(\mathscr{L}_n\mathscr{V})(u_n) = 0$ when the H^s-norm of u_n is smaller that R, we have to consider two cases: when the H^s-norm of u_n is in $[R, 2R]$ or in $(2R, +\infty)$.

- If $R \leq \|u_n\|_s \leq 2R$, from (15) we estimate

$$\begin{aligned}(\mathscr{L}_n\mathscr{V})(u_n) \\ &\leq |\alpha| K I'(\|u_n\|_s)\|u_n\|_{L^\infty}^{2\sigma}\|u_n\|_s + \tfrac{1}{2}I''(\|u_n\|_s)\|P_n\phi(u_n)\|_s^2 + I'(\|u_n\|_s)\frac{\|P_n\phi(u_n)\|_s^2}{\|u_n\|_s} \\ &\lesssim I'(\|u_n\|_s)\|u_n\|_s^{2\sigma+1} + \tfrac{1}{2}I''(\|u_n\|_s)\|\phi(u_n)\|_s^2 + I'(\|u_n\|_s)\frac{\|\phi(u_n)\|_s^2}{R}.\end{aligned}$$

Since $I'(\|u_n\|_s)$, $I''(\|u_n\|_s)$ are continuous, they are bounded when $R \leq \|u_n\|_s \leq 2R$; also $\|\phi(u_n)\|_s$ is bounded when $R \leq \|u_n\|_s \leq 2R$ in virtue of Assumption 16(i).

Therefore

$$\sup_{n\in\mathbb{N}} \sup_{0\le \|u_n\|_s \le 2R} (\mathscr{L}_n \mathscr{V})(u_n) < \infty.$$

• If $\|u_n\|_s > 2R$, then $l(\rho) = \log \rho$. So $l'(\|u_n\|_s) = \frac{1}{\|u_n\|_s}$ and $l''(\|u_n\|_s) = -\frac{1}{\|u_n\|_s^2}$.

Hence, from estimate (15) we infer

$$(\mathscr{L}_n \mathscr{V})(u_n) \le |\alpha| K \|u_n\|_{L^\infty}^{2\sigma} + \frac{1}{2}\frac{\|P_n\phi(u_n)\|_s^2}{\|u_n\|_s^2} - \frac{[\Re(u_n, P_n\phi(u_n))_s]^2}{\|u_n\|_s^4}.$$

Thanks to H5, this quantity is finite if we choose $R = \frac{r}{2}$. Hence

$$\sup_{n\in\mathbb{N}} \sup_{\|u_n\|_s \ge 2R} (\mathscr{L}_n \mathscr{V})(u_n) < \infty.$$

Combining all the above cases we conclude that $(\mathscr{L}_n\mathscr{V}(u_n))$ is bounded in H_n, uniformly in $n \in \mathbb{N}$, that is there must exist a positive constant C - independent of k and n - such that

$$\mathscr{L}_n\mathscr{V} \leq C \quad \text{for any } n \in \mathbb{N}. \tag{17}$$

Hence

$$\mathscr{V}(u_n(t\wedge\tau_{n,k})) \leq \mathscr{V}(u_n(0))+Ct+\int_0^{t\wedge\tau_{n,k}} \mathscr{V}'(u_n(r))[\phi(u_n(r))]\,\mathrm{d}W(r).$$

Taking the expected value on both sides of the above estimate we get (12), from which we infer that $\tau_n = +\infty$ $\mathbb{P}$-a.s., that is, the solution is defined at any time $t \geq 0$.

from Galerkin to the full NLS eq

We work on a finite time interval $[0, T]$, for arbitrary $T > 0$; fix $s' \in (\frac{d}{2}, s)$ and set

$$Z_T := C([0, T]; H^{s'}) \cap C_w([0, T]; H^s).$$

Use the a priori estimates

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\|u_n\|_{L^\infty(0,T;H^s)} \geq C\right) \leq \delta.$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\|u_n\|_{C^{0,\beta}([0,T];H^{-s-1})} \geq C\right) \leq \delta$$

to get that the sequence of the laws of u_n is tight in Z_T.
In metric spaces, one can apply Prokhorov Theorem and Skorohod Theorem to obtain convergence from tightness. Since the space Z_T is a locally convex space, one uses the Jakubowski generalization to non-metric spaces.

Proposition

Assume $\sigma \in \mathbb{N}, s > s' > \frac{d}{2}$ *and H1-H5.*
Then for every $u^0 \in H^s$ *there exists a martingale solution to* (2) *defined on the time interval* $[0, +\infty)$*, with* $\widetilde{\mathbb{P}}$*-a.a. paths in* $C([0, +\infty); H^{s'}) \cap C_w([0, +\infty); H^s)$.

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Then for every $u^0 \in H^s$ there exists a martingale solution to (2) defined on the time interval $[0, +\infty)$, with $\widetilde{\mathbb{P}}$-a.a. paths in $C([0, +\infty); H^{s'}) \cap C_w([0, +\infty); H^s)$.

Exploiting the mild formulation of NLS equation we infer that the solution process has $\tilde{\mathbb{P}}$-a.s. paths in $C([0, \infty); H^s)$.

Proposition

The martingale solution, given in Proposition 2, has a.a. trajectories in $C([0, +\infty); H^s)$.

Finally we prove pathwise uniqueness of the martingale solutions.

Proposition

Assume $\sigma \in \mathbb{N}, s > s' > \frac{d}{2}$ *and H1-H5. Then, the pathwise uniqueness holds.*

For what concerns the problem of existence and uniqueness of invariant measures, this is in general a quite challenging problem for the stochastic NLS equation. By slightly strengthen the assumptions on the diffusion coefficient ϕ and by modifying the Lyapunov function, the same argument used to prove the existence of global solutions, is adapted first to prove the **existence** of invariant measures supported on $H^s(\mathbb{T}^d)$. Then, when $\phi(u) = f(u)u$ for suitable f, we prove that the zero solution is a global attractor so that $\mu = \delta_0$ is the **unique** invariant measure. Roughly speaking, the (even stronger) superlinear noise coefficient "forces the dynamics" to converge to the zero solution, which is an equilibrium of the system.
As far as we know, this is the first result proving the existence and uniqueness of the invariant measure (as well as some stability results) for the NLS on $\mathbb{T}^d$ for an arbitrary large power of the nonlinearity.

To prove the existence of invariant measures μ we need to strengthen Assumption H5

H5 There exist $r > 1$ and $B \in \mathbb{R}$ such that

$$|\alpha| K \|u\|_{L^\infty}^{2\sigma} + \frac{1}{2} \frac{\|\phi(u)\|_s^2}{\|u\|_s^2} \le \frac{[\Re\big(u, \phi(u)\big)_s]^2}{\|u\|_s^4} + B \qquad \forall\ u \in \mathbb{B}^c_{r,H^s}$$

where K is the constant appearing in (3)

as follows

H5' There exist $p \in (0,1)$, $r > 1$ and $B < 0$ such that

$$|\alpha| K \|u\|_{L^\infty}^{2\sigma} + \frac{1}{2} \frac{\|\phi(u)\|_s^2}{\|u\|_s^2} \le (1 - \frac{p}{2}) \frac{[\Re\big(u, \phi(u)\big)_s]^2}{\|u\|_s^4} + B \qquad \forall\ u \in \mathbb{B}^c_{r,H^s}$$

To get uniqueness we strengthen once more.

H5'' There exists $f : H^s \to \mathbb{C}$ such that $\phi(u) = f(u)u$. Moreover, there exist $p \in (0,1)$ and $B < 0$ such that

$$|\alpha| K \|u\|_{L^\infty}^{2\sigma} + \frac{1}{2}|f(u)|^2 \leq (1 - \frac{p}{2})[\Re f(u)]^2 + B, \qquad \forall\ u \in H^s$$

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The latter relation corresponds to

$$|\alpha|K\|u\|_{L^\infty}^{2\sigma} + \frac{1}{2}[\Im f(u)]^2 \le (\frac{1}{2} - \frac{p}{2})[\Re f(u)]^2 + B. \tag{18}$$

Statement of the main results

We summarize our main results as follows.

Theorem

Let

$$\sigma \in \mathbb{N}, \qquad s > \frac{d}{2}.$$

If H1-H5' hold, then there **exists** *at least one invariant measure* μ *for equation* (8)*, supported in* H^s*, and*

$$\int_{H^s} \|x\|_s^p \mathrm{d}\mu(x) < \infty,$$

where $p \in (0, 1)$ *is the parameter in condition H5'.*

If H1-H5" hold, then $\mu = \delta_0$ *is the* **unique** *invariant measure for equation* (8) *and the zero solution is exponentially stable.*

The Galerkin approximated NLS equation has a stationary solution (constructed by means of the Krylov-Bogoliubov technique).
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We use as Lyapunov function

$$\mathscr{V}(u) = I(\|u\|_s),$$

where $I : [0, +\infty) \to [a, +\infty)$ is a non-decreasing C^2-function such that

$$\begin{cases} I(\rho) = a, & 0 \le \rho < R \\ I(\rho) = \rho^p, & \rho > 2R, \end{cases} \tag{19}$$

for some $a \in (0, (r)^p)$ and choosing $R = \frac{r}{2}$.

The difference with respect to the function defined in (13) is for large values of ρ (i.e. outside the ball of radius $2R$), where we have

$$l'(\rho) = p\rho^{p-1} \quad \text{and} \quad l''(\rho) = -p(1-p)\rho^{p-2}. \tag{20}$$

Thus we get

$$\frac{1}{T}\int_0^T \mathbb{E}(\|u_n(t)\|_s^p)\,\mathrm{d}t \le \frac{a+M}{p\tilde{B}}.$$

Now use Krylov-Bogoliubov technique ...

As far as the uniqueness of the invariant measure is concerned, we have

Theorem

Assume H1-H5''. Then for any $u^0 \in H^s$

i) *the zero solution is exponentially stable in the p-mean, where $p \in (0,1)$ is the parameter in condition H5'', that is there exist constants $C < \infty$, $\lambda > 0$, such that*

$$\mathbb{E}[\|u(t)\|_s^p] \le Ce^{-\lambda t}\|u^0\|_s^p, \qquad \forall\ t \ge 0;$$

ii) *the zero solution is exponentially stable with probability one, that is, for any $\bar{\lambda} \in (0,\lambda)$, there exists a $\mathbb{P}$-a.s. finite random time τ_0 such that*

$$\|u(t)\|_s^p \le Ce^{-\bar{\lambda} t}\|u_0\|_s^p, \qquad \forall\ t \ge \tau_0, \quad \mathbb{P}-a.s.;$$

iii) *$\mu = \delta_0$ is the unique invariant measure.*

rewriting H5" as

$$|\alpha| K \|u\|_{L^\infty}^{2\sigma} + \frac{1}{2}[\Im f(u)]^2 \leq \frac{1-p}{2}[\Re f(u)]^2 + B, \quad \forall\, u \in H^s$$

we realize that a key role is played by the real part of f, which dominates the intensity of its imaginary part and of the nonlinear term.

Compared to previous results, our paper presents novelties in different directions.

When compared to the deterministic literature, our result can be understood as a regularization (no blow-up) by noise result.

When compared to the stochastic literature, as far as we know, this is the first result providing the existence of a unique global solution in the regular Sobolev space $H^s(\mathbb{T}^d)$, $s > \frac{d}{2}$, for an arbitrary large power of the nonlinear term and for any initial data in $H^s(\mathbb{T}^d)$ (both in the focusing and defocusing case). And providing results on invariant mesures.

Based on the paper with Brzeźniak, Maurelli and Zanella:
Global well posedness and ergodic results in regular Sobolev spaces for the nonlinear Schrödinger equation with multiplicative noise and arbitrary power of the nonlinearity
Discrete Contin. Dyn. Syst. 45 (2025), no. 9, 3217-3257

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