Nash Inequalities and Boundary Behavior of Kinetic Equations

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(University of Arizona)

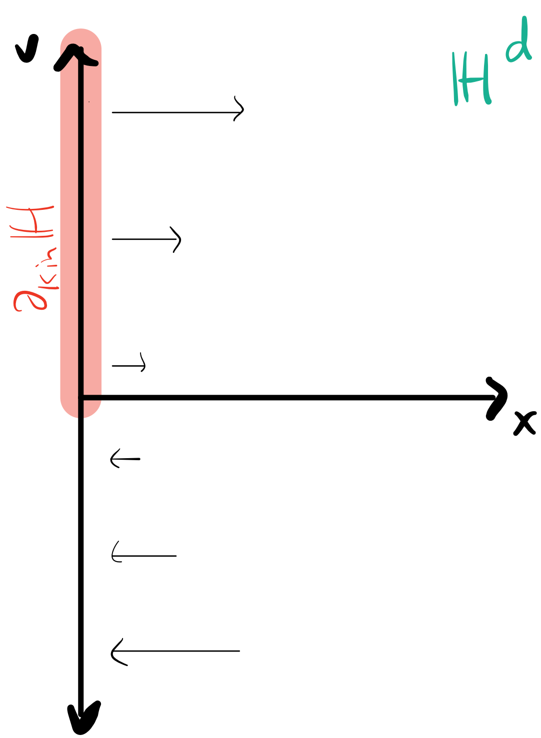
June 23, 2025

⊛ Joint with Lucertini and Wang

The Basics

$$\begin{cases} (\partial_t + v\cdot\nabla_x) f = \Delta_v f & (x,v)\in \mathbb{H}^d \\ f|_{t=0} = f_0 & \\ f|_{\partial_{kin}\mathbb{H}^d} = 0 & \end{cases}$$

f = density of particles ≥ 0
x = locn, v = velocity



Motivation:

* Gas dynamics:
  Landau + Boltzmann equations
* Integrated Brownian Motion

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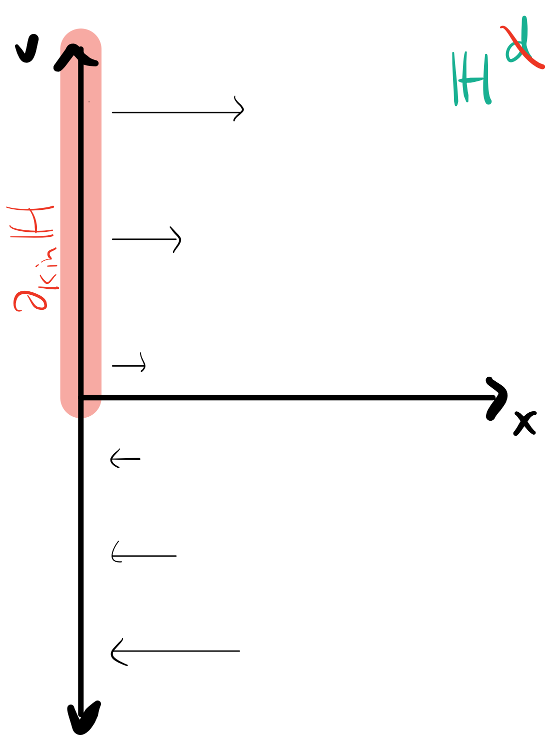

Motivation:

* Gas dynamics:

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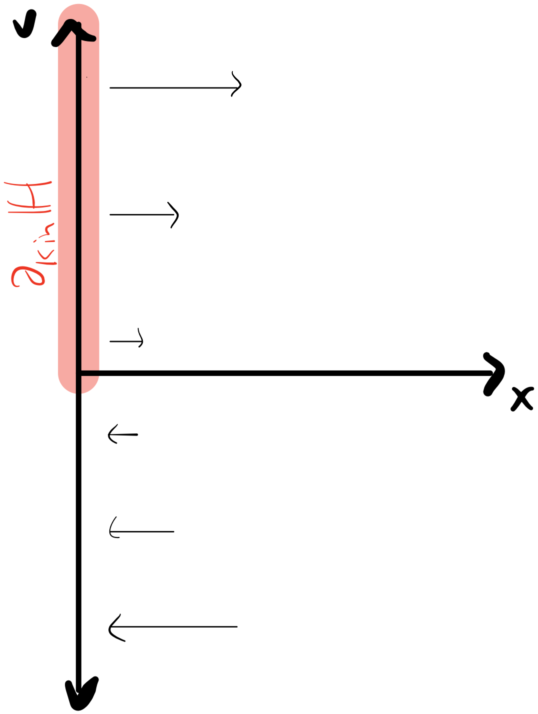
$$\begin{cases} (\partial_t + v\cdot\nabla_x) f = \Delta_v f & x>0,\ v\in\mathbb{R} \\ f|_{t=0} = f_0 & \\ f|_{\partial_{kin}\mathbb{H}} = 0 & x=0,\ v>0 \end{cases}$$

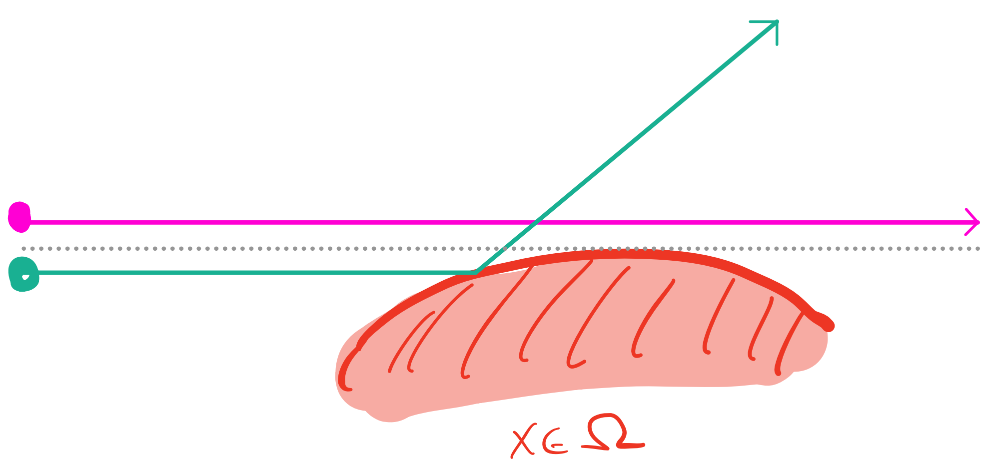
$$f(t,x,v) = \mathbb{E}\left[f_0\left(X_t, V_t\right) \mathbb{1}\{t<\tau\}\right]$$

$\tau =$ First exit time, $X_t = x - \int_0^t V_s\, ds$, $V_t = v + \sqrt{2} B_t$



Main Question: Behavior / regularity of f "near" boundary?

Difficulty: "Grazing set": two particles with similar starting position have diff. long-time behavior



normal at $x\in\partial\Omega$

General: $v\cdot\eta(x) = 0$

Today: $x=0,\ v=0$

The Basics

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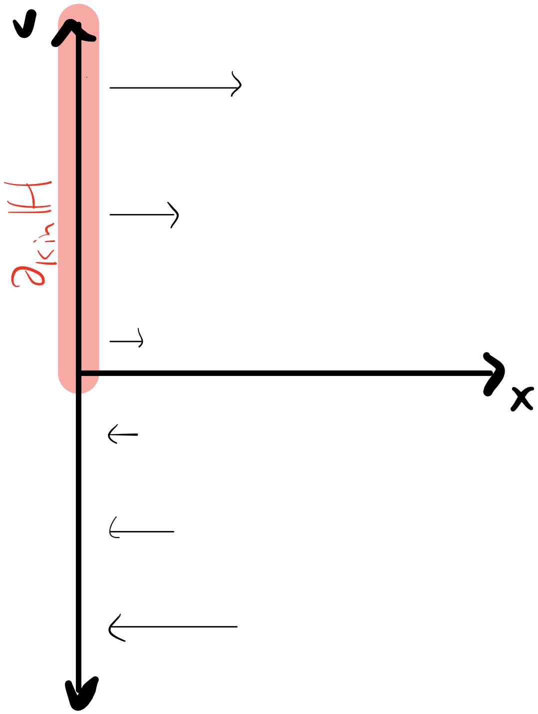

Main Question: Behavior / regularity of f "near" boundary?

Known:

* Interior Estimates (too many to list)

  De Giorgi + Schauder:

  ↳ All solns C^α (some $\alpha>0$)

  ↳ Coeff. $C^{k+\alpha}_{kin} \Rightarrow f \in C^{k+2,\alpha}_{kin}$

* De Giorgi type bdry estimate (irreg. coeff.)

  Silvestre, Zhu

  ↳ $\exists\, \alpha>0$ s.t. $f\in C^\alpha$ up to boundary

  ↳ Silv.: $f \leq (\text{dist to bdry})^k \quad \forall k$

  away from grazing set

* Homog. case: Hwang-Jang-Velázquez

  $f \in C^{1/2-\varepsilon}_{kin} \approx C^{\frac{1}{2}-\varepsilon}_x C^{1/2-\varepsilon}_v \quad \forall \varepsilon$

Nice scaling: only need to understand decay rate near bdry!

The Basics

$$\begin{cases} (\partial_t + v\cdot\nabla_x) f = \Delta_v f & x>0,\ v\in\mathbb{R} \\ f|_{t=0} = f_0 & \\ f|_{\partial_{kin}\mathbb{H}} = 0 & x=0,\ v>0 \end{cases}$$

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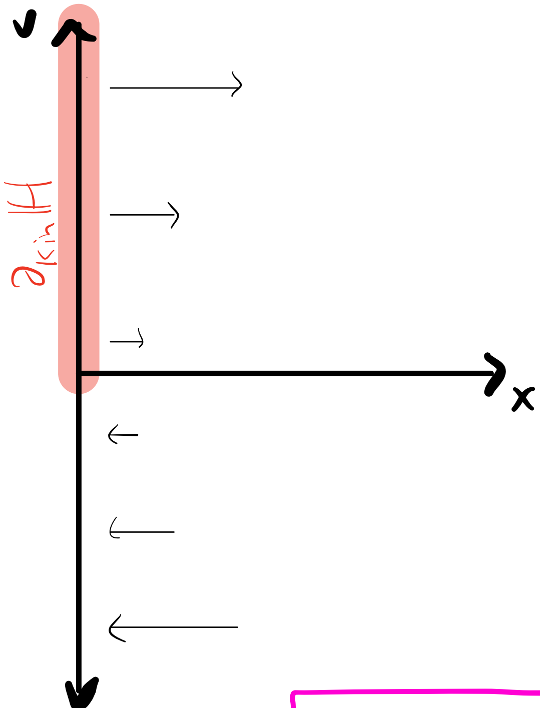

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  (($f \in C^{1/2-\varepsilon}_{kin} \approx C^{1/6-\varepsilon}_x C^{1/2-\varepsilon}_v \quad \forall \varepsilon$

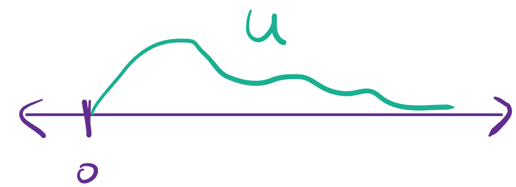
$\boxed{\text{Goal: Make these precise}}$

Note: Sharp!

Groeneboom – Jongbloed – Wellner

A Toy Model

$$\left\{\begin{array}{ll} u_t = u_{xx} & x>0 \\ u(t,0)=0 & \end{array}\right\}$$



Thm: $u(t,x) \lesssim \dfrac{x}{t^{3/2}} \|x u_0\|_{L^1}.$

$$\boxed{\textbf{Lemma:} \quad \int_0^\infty u^2 \lesssim \left(\int_0^\infty x u\right)^{4/5} \left(\int_0^\infty u_x^2\right)^{3/5} \qquad \text{"Bdry Nash Ineq."}}$$

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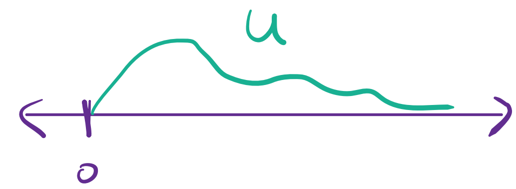

Lemma: $\int_0^\infty u^2 \lesssim \left(\int_0^\infty xu\right)^{4/5} \left(\int_0^\infty u_x^2\right)^{3/5}$ "Bdry Nash Ineq."

Pf: Std Nash $\int_{\sqrt{R}}^\infty u^2 \lesssim \left(\int_{\sqrt{R}}^\infty \frac{x}{\sqrt{R}} u\right)^{4/3} \left(\int_{\sqrt{R}}^\infty u_x^2\right)^{1/3}$

Poincare $\int_0^{\sqrt{R}} u^2 \lesssim R \int_0^{\sqrt{R}} u_x^2$.

Optimize in R!

Crucial Decomposition:

* Nash inequality where steady state x bdd from below
* Poincaré near bdry

A Toy Model

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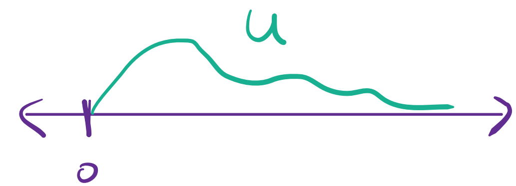

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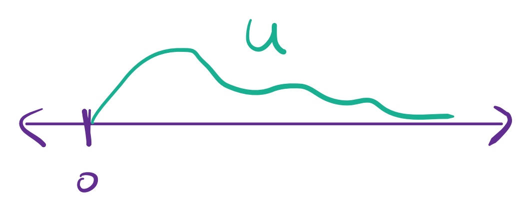

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"Bdry Nash Ineq."

Pf: $\bullet\ \partial_t \int_0^\infty xu = 0 \qquad \int_0^\infty xu = \int_0^\infty x u_0$

$$\bullet\ \frac{1}{2}\partial_t \int_0^\infty u^2\, dx = -\int_0^\infty u_x^2 \lesssim - \frac{\left(\int_0^\infty u^2\right)^{5/3}}{\left(\int_0^\infty x u_0\right)^{4/3}}$$

Diff. ineq. $\dot{X} \lesssim -X^{5/3}$

$$\Rightarrow \left(\int_0^\infty u^2\, dx\right)^{1/2} \lesssim \frac{\int_0^\infty x u_0}{t^{3/4}}$$

$\Rightarrow S_t : L^1_x \to L^2$ (soln op) $\qquad \|S_t\| \lesssim 1/t^{3/4}$

adj op: soln op.

$S_t^* : L^2 \to L^\infty_{1/x} \qquad \|S_t^*\| \lesssim 1/t^{3/4}$

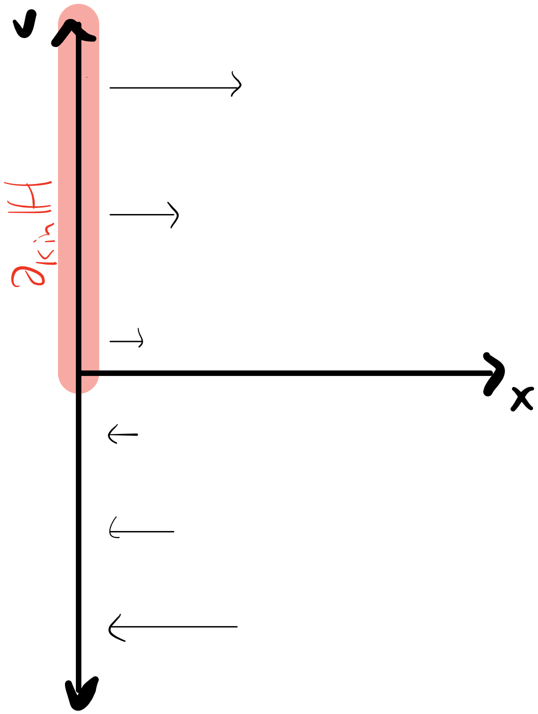
$$\frac{u(t,x)}{x} \overset{u(t) = S^*_{t/2} u(t/2)}{\lesssim} \frac{\left(\int_0^\infty u^2(t/2)\, dx\right)^{1/2}}{t^{3/4}} \overset{u(t/2) = S_{t/2} u_0}{\lesssim} \frac{1}{t^{3/4}}\left(\frac{\int_0^\infty x u_0}{t^{3/4}}\right)$$

Ingredients:

* Conserved quantity (steady state of adj. eqn)
* Energy equality
* Bdry Nash

Back to Kinetic Fokker-Planck

$$\begin{cases} (\partial_t + v\cdot\nabla_x) f = \Delta_v f & x>0,\ v\in\mathbb{R} \\ f|_{t=0} = f_0 \\ f|_{\partial_{kin}\mathbb{H}} = 0 & x = 0,\ v>0 \end{cases}$$



Ingredient #1: Conserved Quantity

↳ Steady State of adjoint eqn

$$(\partial_t - v\cdot\nabla_x - \Delta_v) f = 0$$

$\varphi^*(x,v) = \varphi(x,-v)$ solves this!

$$\Rightarrow \int_{\mathbb{H}} f\varphi^* = \int_{\mathbb{H}} f_0\,\varphi^*$$

Ingredient #2: Energy Equality

$$\frac{1}{2}\partial_t \int_{\mathbb{H}} f^2 = \int_{\mathbb{H}} (\Delta_v f - v\cdot\nabla_x f) f = -\int_{\mathbb{H}} |\nabla_v f|^2 + \int_{\partial\mathbb{H}} \underbrace{v f(x=0,v)}_{\begin{cases} =0 & v>0 \\ <0 & v<0 \end{cases}} dv$$

Ingredient #3: Boundary Nash

If true, then:

$$\int_{\mathbb{H}} f^2 \lesssim \frac{\left(\int_{\mathbb{H}} \varphi^* f_0\right)^2}{t^{2d+1/2}}$$

Back to Kinetic Fokker-Planck

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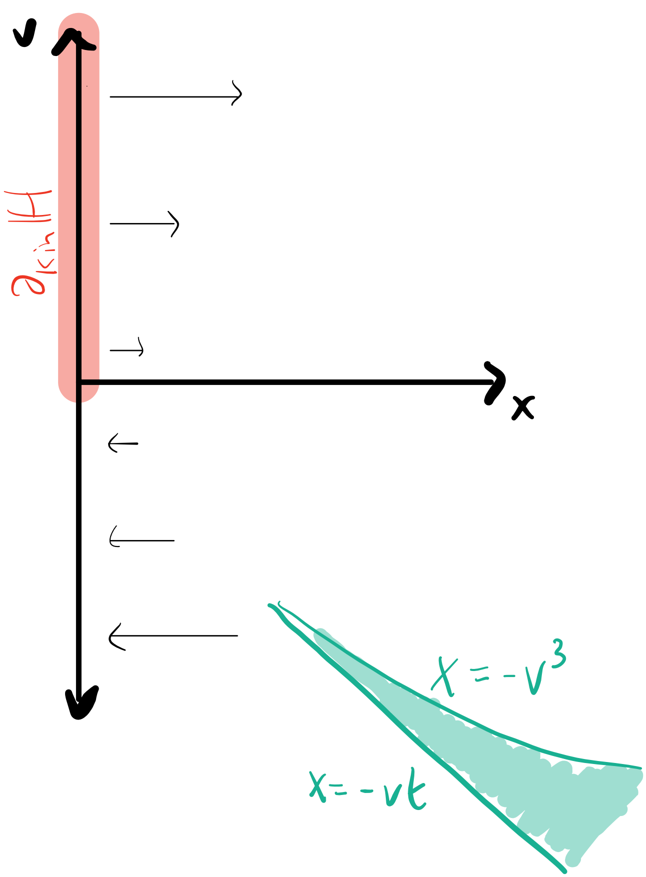

Ingredient #1: Conserved Quantity $\int_{\mathbb{H}} \varphi^* f = \int_{\mathbb{H}} \varphi^* f_0$

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Ingredient #3: Boundary Nash

Why not true? "Isolated Region"

↳ B_t moves by $O(\sqrt{t})$ in v

↳ X_t moves by $O(t^{3/2}) + O(t v_0)$ in x

So: • If $-v_0 \gg \sqrt{t}$ then $B_t + v_0 \approx v_0$

• If $x_0 \gg -v_0^3 \gg v_0 t$ then $X_t \approx x_0$

$\Rightarrow$ Doesn't explore space

$\Rightarrow$ Doesn't feel bdry

But then why is φ^* so small there?

Back to Kinetic Fokker-Planck

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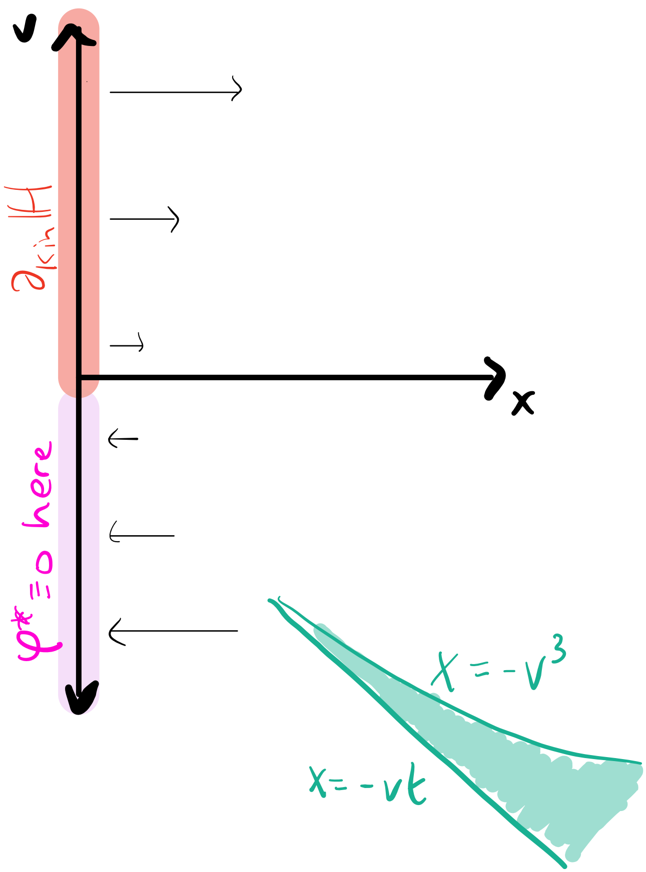

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But then why is φ^* so small here?

τ = first exit time

$$-vt \ll x = -v\tau + O(\tau^{3/2})$$

① $\tau \approx \frac{x}{|v|}$ (uses $x \ll -v^3$)

② Diffusion negligible : transport determines behavior

③ $\tau \gg t$

φ^* "feels" bdry but on longer time scale than f sees so far

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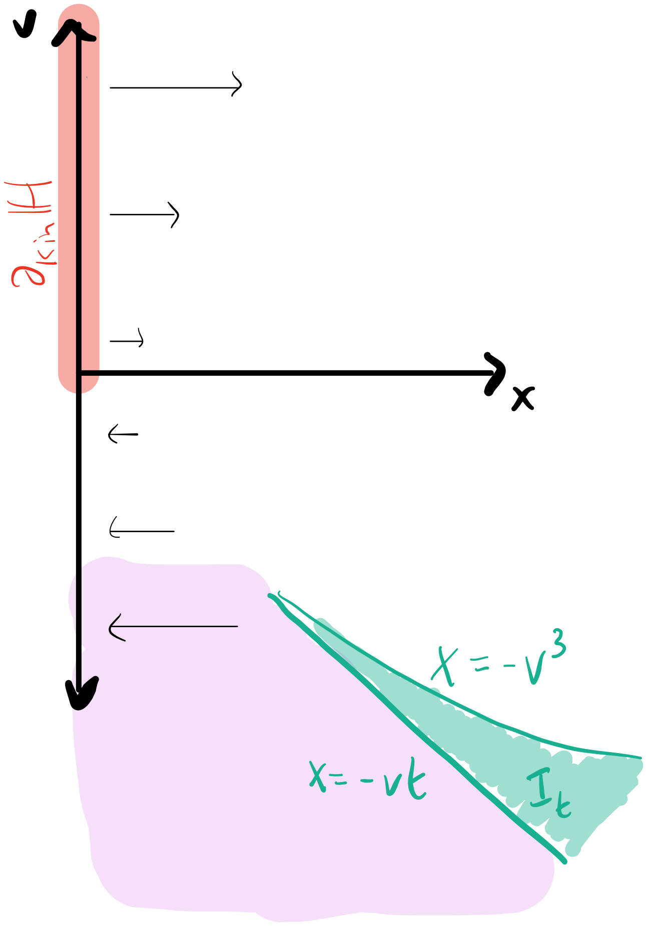

What happens in isolated region? (I_t)

↳ On top: $\varphi^t \gtrsim \mathcal{O}(t^{1/4})$ — Desired lower bd!

↳ on left: particles leave by transport "good sign"

$$\Rightarrow \mu_t^*(x,v) = \begin{cases} 1 & \text{isolated region} \\ e^{\frac{vt}{cx}} & -v \gg \sqrt{t},\ x \ll -tv \\ 0 & \text{elsewhere} \end{cases}$$

$$\Rightarrow (\partial_s - v\nabla_x)\mu_t^* - \Delta_v \mu_t^* \lesssim \frac{1}{t^{5/4}}\varphi^t \quad \forall s \le t$$

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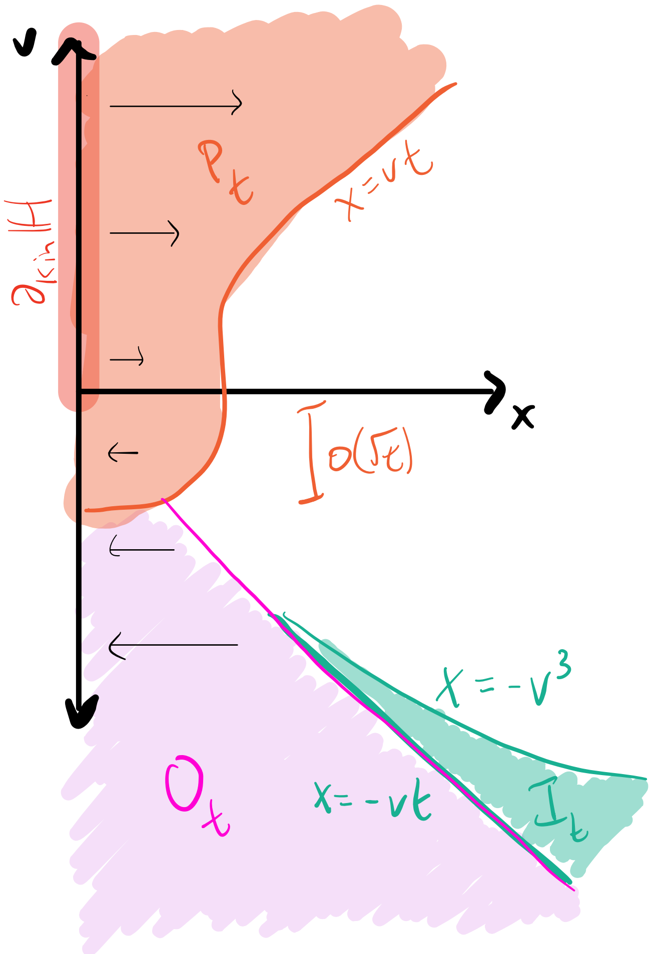

Incoming Poincaré Region (P_t)

Kinetic dist. to $\partial_{kin}\mathbb{H}$ $\lesssim \sqrt{t}$

$$\int_{P_t} f^2 \lesssim t\,\|f\|^2_{H^1_{kin}}$$

Outgoing Poincaré Region (O_t)

$x \le -vt$ (leaves domain by transport)

$$\int_{O_t} f^2 \lesssim t\,\|f\|^2_{H^1_{kin}} + t\int_{-\infty}^{0} v\, f(x=0,v)^2\,dv$$

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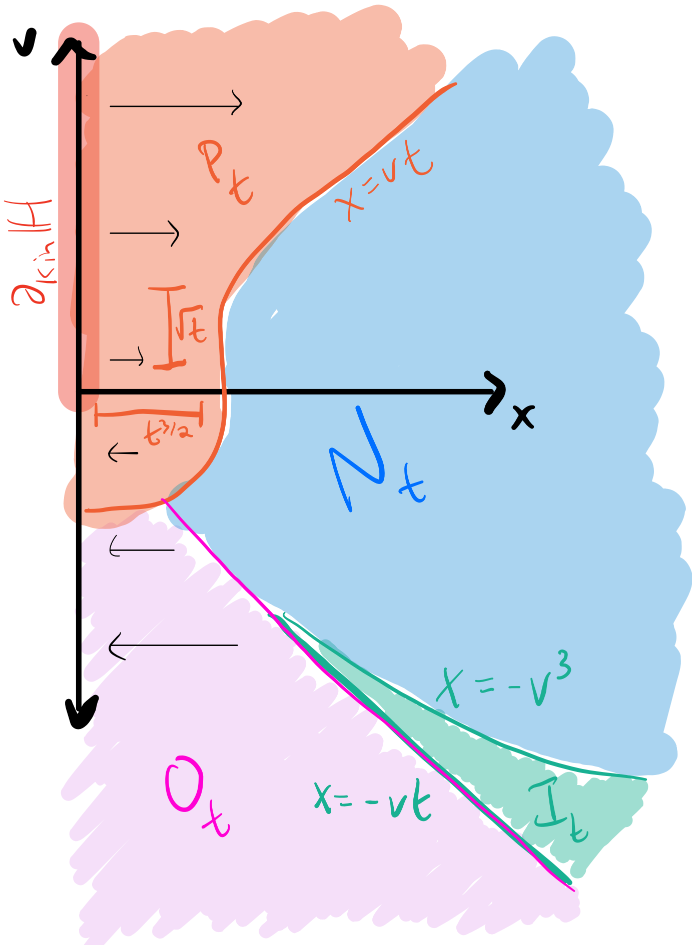

Nash Region (N_t): the rest of the domain

$$\int_{N_t\cup I_t} f^2 \lesssim t\,\|f\|_{H^1_{kin}} + \frac{1}{t^2}\left(\int_{N_t\cup I_t} f\,dx\,dv\right)^2$$

$$\varphi^*(x,v) = \begin{cases} \dfrac{x}{v^{5/2}} e^{-\frac{v^3}{9x}} & 0\le x\le v^3 \\ x^{1/6} & |v|^3\le x \\ \sqrt{|v|} & 0\le x\le -v^3 \end{cases}$$

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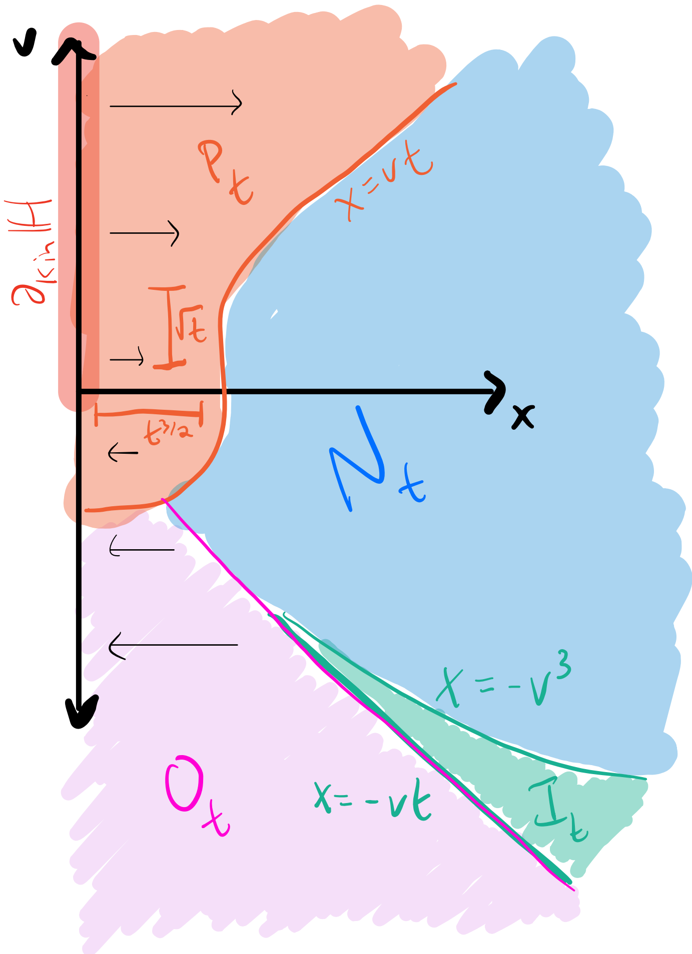

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$$\int_{N_t \cup I_t} f^2 \lesssim t \, \|f\|_{H^1_{kin}} + \frac{1}{t^2}\left(\int_{N_t} \frac{\varphi^*}{t^{1/4}} f_0 \, dx\, dv\right)^2 + \frac{1}{t^2}\left(\int \mu_t^* f\right)^2$$

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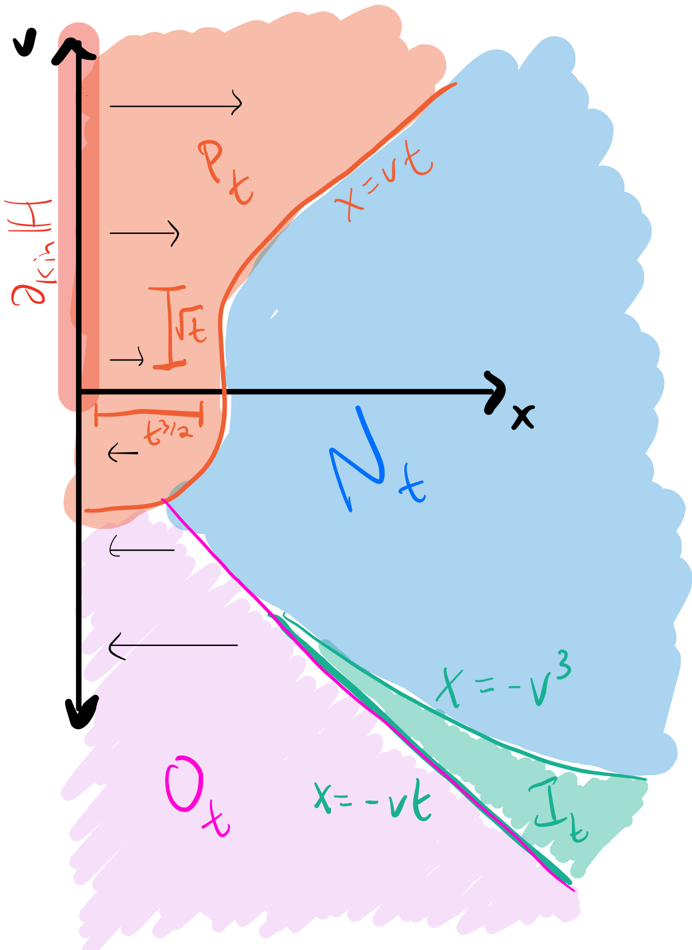

Putting it all together!

$$\int f\mu_t^* \lesssim \int \mu_t^* f_0 + \frac{1}{t^{1/4}}\int \varphi^* f_0$$

$$\int_{\mathbb{H}} f^2 = \begin{cases} \displaystyle\int_{P_t} f^2 \lesssim t\,\|f\|^2_{H^1_{kin}} \\ \displaystyle\int_{O_t} f^2 \lesssim t\,\|f\|^2_{H^1_{kin}} + t\int_{-\infty}^{0} v\, f(x=0,v)^2 \\ \displaystyle\int_{N_t\cup I_t} f^2 \lesssim t\,\|f\|^2_{H^1_{kin}} + \frac{1}{t^2}\left(\int_{N_t} \frac{\varphi^*}{t^{1/4}} f_0\right)^2 + \frac{1}{t^2}\left(\int \mu_t^* f\right)^2 \end{cases}$$

$$\frac{1}{2}\partial_t \int_{\mathbb{H}} f^2 \approx -\|f\|^2_{H^1_{kin}} - \int_{\partial\mathbb{H}} v\, f(x=0,v)^2$$
$$\lesssim -\frac{1}{t}\int_{\mathbb{H}} f^2 + \frac{1}{t^{7/2}}\left(\int_{\mathbb{H}} \varphi^* f_0\right)^2 + \frac{1}{t^{7/2}}\left(t^{1/4}\int_{\mathbb{H}} \mu^* f_0\right)^2$$

$$\Rightarrow \quad \left(\int_{\mathbb{H}} f^2\right)^{1/2} \lesssim \frac{\int_{\mathbb{H}} \varphi^* f_0 + t^{1/4}\int_{\mathbb{H}} \mu_t^* f_0}{t^{5/4}}$$

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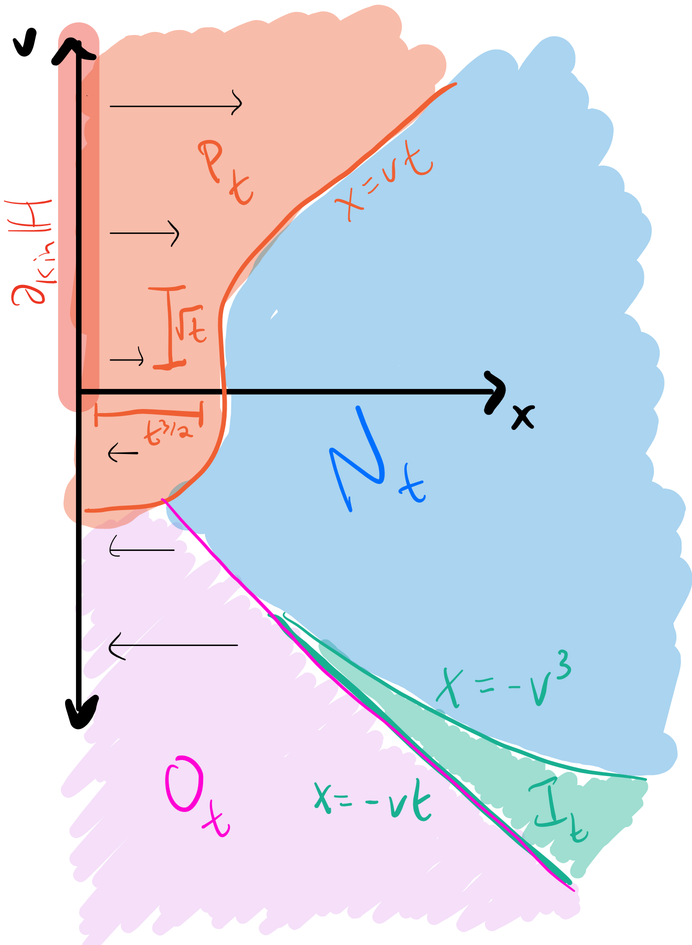

$$\boxed{\textbf{Thm:}\quad f \lesssim \left(\frac{\varphi + t^{1/4}\mu_t}{t^{2d+1/2}}\right)\left[\int_{\mathbb{H}^d} \varphi^* f_0 + t^{1/4}\int_{\mathbb{H}^d}\mu_t^* f_0\right]}$$

<u>Special case:</u> $f_0 = 0 \quad \forall\ v \leq -L$

- $t \gg L^2 \Rightarrow \mu_t^* f \equiv 0.$

$$\Rightarrow f \lesssim \left(\frac{\varphi + t^{1/4}\mu_t}{t^{2d+1/2}}\right)\int_{\mathbb{H}^d}\varphi^* f_0$$

- For $v \ll \sqrt{t}$:

$$\Rightarrow f \lesssim \frac{\varphi}{t^{2d+1/2}}\int_{\mathbb{H}^d}\varphi^* f_0$$

Just like heat eqn!

$$\mu_t(x,v) = \mu_t^*(x,-v).$$

$$\approx e^{-\frac{tv}{cx}} \quad \text{if } 0\leq x\leq vt,\ v\gtrsim\sqrt{t}$$

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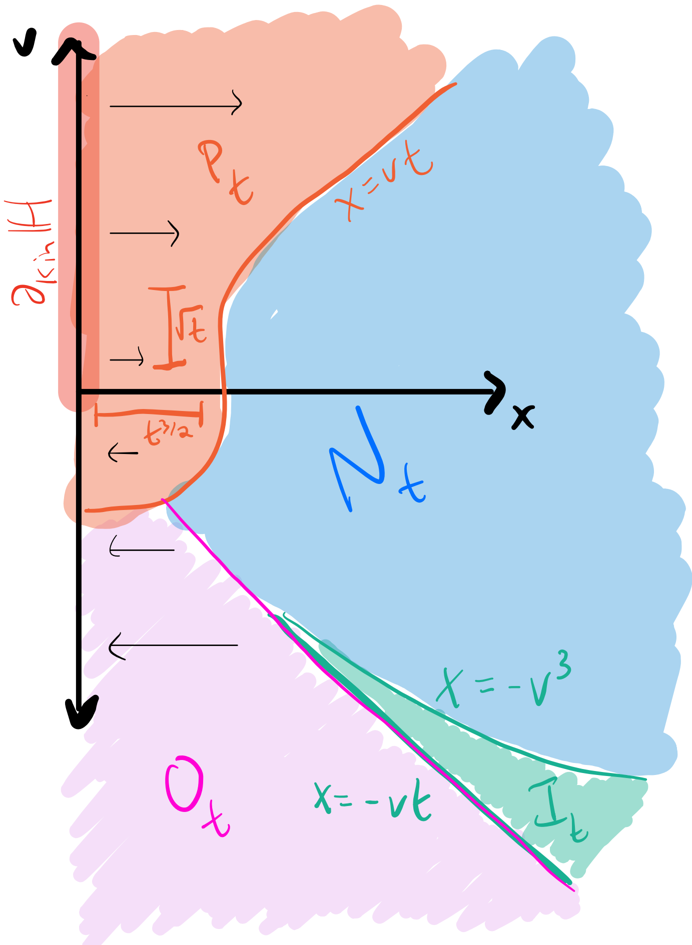


Thm:

$$\boxed{f \lesssim \left(\frac{\varphi + t^{1/4}\mu_t}{t^{2d+1/2}}\right)\left[\int_{\mathbb{H}^d}\varphi^* f_0 + t^{1/4}\int_{\mathbb{H}^d}\mu_t^* f_0\right]}$$

Features:

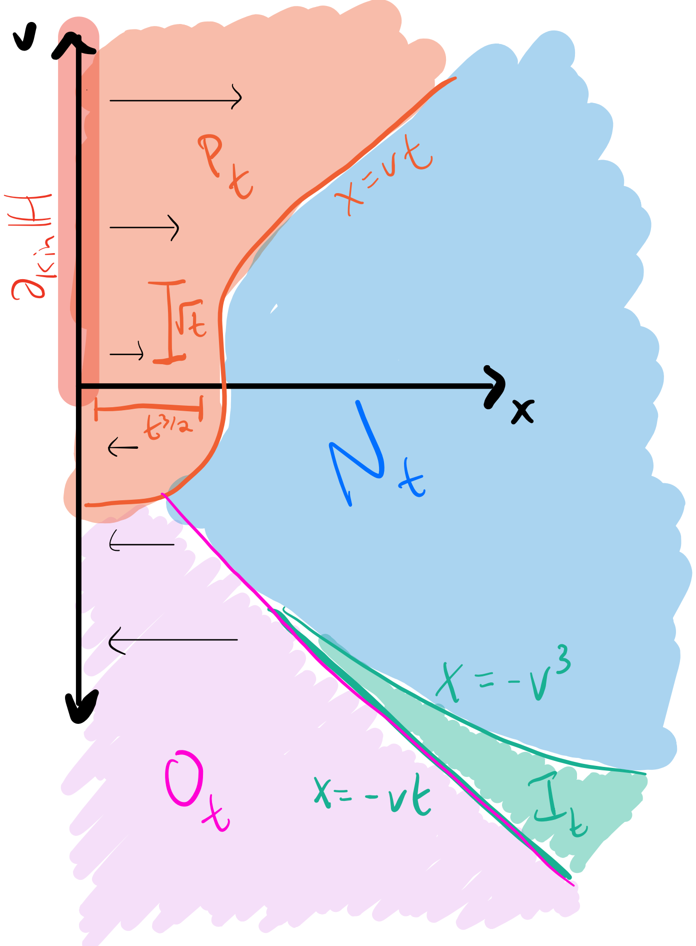
- $0 \lesssim x \lesssim v^3$: $f(t,x,v) \approx_t \begin{cases} \frac{x}{v^{5/2}} e^{-\frac{v^3}{9x}} & v \lesssim \sqrt{t} \\ e^{-\frac{vt}{cx}} & v \gtrsim \sqrt{t} \end{cases}$ (Correct time decay)
- $0 \lesssim |v|^3 \lesssim x \to 0$: $f(t,x,v) \approx_t x^{1/6}$
- $0 \lesssim x \lesssim -v^3 \to 0$: $f(t,x,v) \approx_t \sqrt{|v|}$

Hence $f \in C^{1/2}_{kin}$ and decays exp at bdry away from grazing set

- Flexible! Requires only one "good" soln

$\mu_t(x,v) = \mu_t^*(x,-v)$.

$\approx e^{-\frac{tv}{cx}}$ if $0 \le x \le vt$, $v \gtrsim \sqrt{t}$



Thanks For Your Attention!

$\tau =$ exit time $\gg 1$ $\qquad -vt \ll x \ll -v^3$

$$-vt \ll X = -\int_0^{\tau} V_s\, ds = -V\tau + O(\tau^{3/2})$$

error

① $\tau^{3/2} \approx \left(\frac{x}{|V|}\right)^{3/2} \ll X$

② $\tau \gg t$ $\qquad$ Don't "see bdy" at t

③ $O(\tau^{1/2}) \ll |V|$ $\qquad$ See bdy before diffuse away

(makes φ^* small)

What happens if we take $x \approx \tau^{3/2}$?

i.e. $\sqrt{\tau} \gg |V|$.

Then $x \gg |V|^3$ $\cancel{\ast}$