Quantitative approximation of the Navier-Stokes-Vlasov-Fokker-Planck System by stochastic particles systems

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Program.

- Macroscopic model-Navier-Stokes-Vlasov-Fokker-Planck
- Microscopic model.
- About Week solution/bounded solution, and regularity of Navier-Stokes-Vlasov-Fokker-Planck.
- Hypothesis and results.
- Some idea of the proof.
- Some future problems.
- Main bibliography.

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Navier-Stokes-Vlasov-Fokker-Planck.

This work is concerned with the mathematical description of the motion of a dispersed phase of small particles (e.g. a spray or an aerosol) flowing in a surrounding incompressible homogeneous fluid. The cloud of particles (resp. the fluid) is described by its distribution function f in the phase space (resp. its velocity u and its pressure p).

Navier-Stokes-Vlasov-Fokker-Planck.

- The mathematical analysis of the coupled system (1) in dimension $d = 2, 3$ has received much attention in the past ears.

$$\begin{cases} \partial_t u_t(x) - \Delta u_t(x) + (u_t \cdot \nabla)[u_t](x) + \nabla p_t(x) \\ = -\int_{\mathbb{R}^d} (u_t(x) - v') F(t, x, v') dv', \quad t > 0, \ x \in \mathbb{T}^d, \\ \operatorname{div}_x u_t(x) = 0 \\ u(0, x) = u_0(x), \quad x \in \mathbb{T}^d \\ \partial_t F(t, x, v) + v \cdot \nabla_x F(t, x, v) + \operatorname{div}_v((u_t(x) - v) F(t, x, v)) = \\ \frac{\sigma^2}{2} \Delta_v F(t, x, v), \ (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \\ F(0, x, v) = F_0(x, v), \quad x \in \mathbb{T}^d \times \mathbb{R}^d, \end{cases} \tag{1}$$

where p is the pressure to ensure the divergence free condition in the first equation, u_0 is the initial vector fluid velocity with values in $\mathbb{R}^d$, and F_0 is the initial density of particles with respect to the position variable $x \in \mathbb{T}^d$ and to the velocity variable $v \in \mathbb{R}^d$.

Navier-Stokes-Vlasov-Fokker-Planck.

- Our main interest : **quantitative** approximation of (1) from microscopic model.
- We consider the microscopic model introduced in F. Flandoli, M. Leocata, M., C. Ricci, The Navier-Stokes-Vlasov-Fokker-Planck System as a Scaling Limit of Particles in a Fluid. J. Math. Fluid Mech., 2021.

Microscopic model.

- We introduce the mollified delta Dirac function

$$\delta^N_{X_t^{i,N}}(x) = \vartheta^{1,N}(x - X_t^{i,N}).$$

We introduce the particle system as a coupling between a Navier-Stokes PDE with drag force and a system of Stochastic Differential Equations driven by independent Brownian motions.

$$\begin{cases} \partial_t u_t^N(x) - \Delta u_t^N(x) + (u_t^N \cdot \nabla)\left[\boldsymbol{\chi}_A(u_t^N)\right](x) + \nabla p_t^N(x) \\ = -\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\chi}_A(u_t^N(X_t^{i,N})) - V_t^{i,N})\delta^N_{X_t^{i,N}}(x), \\ \mathrm{div}\, u_t^N = 0 \\ u^N(0,x) = u_0^N(x), \quad x \in \mathbb{T}^d, \\ dX_t^{i,N} = V_t^{i,N}\mathrm{d}t \\ dV_t^{i,N} = (\boldsymbol{\chi}_A(u_t^N(X_t^{i,N})) - V_t^{i,N})\mathrm{d}t + \sigma \mathrm{d}B_t^i \end{cases} \tag{2}$$

harmonise $u(t,\cdot)$ and $u_t(\cdot)$ where N is the number of particles and $(X_t^{i,N}, V_t^{i,N})$ for $1 \leq i \ \leq N$ are position and velocity of the i^{th} particle. $\boldsymbol{\chi}_A$ is a smooth cut-off function controlled by nonnegative real parameter A that will be fixed later.

Microscopic model.

- They show the convergence of the microscopic to macroscopic system for $d = 2$ and $\sigma > 0$.

Our contribution.

- We quantify the distance between the microscopic and macroscopic model in suitable distance.
- We include $d = 2, 3$, $\sigma > 0$ and $\sigma = 0$.

Other models.

- Goudon, T., Jabin, P.-E., Vasseur, A., Hydrodynamic limit for the Vlasov–Navier–Stokes equations. Part I: light particles regime, Indiana Univ. Math. J. 53, 1495–1515 , 2004.
- Goudon, T., Jabin, P.-E., Vasseur, A., Hydrodynamic limit for the Vlasov–Navier–Stokes equations. Part II: fine particles regime. Indiana Univ. Math. J. 53.

Definition of Week solution .

We say that (u, F) is a weak solution to the Navier-Stokes-Vlasov-Fokker-Planck system (1) if

$$u \in L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2([0,T];H^1(\mathbb{T}^d)),\ \operatorname{div} u = 0$$
$$F(t,x,v) \geq 0,\ \forall\ (t,x,v) \in (0,T) \times \mathbb{T}^d \times \mathbb{R}^d,$$
$$F \in L^\infty(0,T;L^\infty(\mathbb{T}^d \times \mathbb{R}^d) \cap L^1(\mathbb{T}^d \times \mathbb{R}^d)),$$
$$|v|^2 F \in L^\infty(0,T;L^1(\mathbb{T}^d \times \mathbb{R}^d)),$$

satisfies, for all $\phi \in C^\infty([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$ with compact support in v, such that $\phi(T,\cdot,\cdot) = 0$,

Definition of week solution.

$$-\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} F\big[\partial_t \phi + v \cdot \nabla_x \phi + (u - v) \cdot \nabla_v \phi - \frac{\sigma^2}{2} \Delta_v \phi\big] dxdvds$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} F_0 \phi(0, x, v) dxdv$$

and, for all $\psi \in C^\infty([0, T] \times \mathbb{T}^d)$ with $\operatorname{div} \psi = 0$ and $\psi(T, \cdot) = 0$,

$$-\int_0^T \int_{\mathbb{T}^d} u \partial_t \psi dxds + \int_0^T \int_{\mathbb{T}^d} u \cdot \nabla u \psi dxds + \int_0^T \int_{\mathbb{T}^d} \nabla u : \nabla \psi dxds$$

$$= -\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} F(u - v) \psi dxdvds + \int_{\mathbb{T}^2} u_0 \psi(0, x) dx.$$

Definition of week solution.

- There exist global in time week solution $d = 2$ and $d = 3$. L Boudin, L Desvillettes, C Grandmont, A Moussa, Differential Integral Equations, 2009. C. Yu - Journal de Mathématiques Pures et Appliquées, 2013.

- Uniqueness week solution $d = 2$, Daniel Han-Kwan, E. Miot, A. Moussa, I. Moyano, Rev. Mat. Iberoam, 2020.

Why don't we work with weak solutions?.

- $\mu * \vartheta^N u - \mu u * \vartheta^N$, in the setting of quantitative estimation, we need to know the rate convergence of the commutator, for this we NEDD the regularity of u.

- $d = 2$ the uniqueness solution u, by Chemin-Lerner, has only log-Lipschitz regularity

$$|u(t,x) - u(t,y)| \leq \gamma(t)|x-y|(log|x-y|-1)^{\eta},$$

But $\gamma(t) \in L^1$!!!!

Bounded .

During this work we assume that

$$u \in L^{\infty}([0,T] \times \mathbb{R}^d).$$

Under suitable hypothesis this is true

- $\sigma > 0$ and $d = 2$, F. Flandoli, M. Leocata, M., C. Ricci, Fluid. J. Math. Fluid Mech., 2021.
- $\sigma > 0$ and $d = 3$, T. Goudon, L. He, A. Moussa, and P. Zhang, SIAM Journal on Mathematical Analysis, 2010.

Bessel Sapce.

Let $\mathcal{D}(\mathbb{T}^d)$ be the collection of all infinitely differentiable functions on $\mathbb{T}^d$. Then $\mathcal{D}'(\mathbb{T}^d)$ stands for the topological dual of $\mathcal{D}(\mathbb{T}^d)$. We denote the Fourier coefficients of $u \in \mathcal{D}'(\mathbb{T}^d)$ by $\hat{u}(k) := \frac{1}{(2\pi)^{d/2}} u(e^{2i\pi\langle k,\cdot\rangle})$notation. As there should be no risk of confusion, we will use the same notations for the Bessel norm and Bessel operator on the torus as in the whole space. Hence for any $\gamma \in \mathbb{R}$, we define the Bessel potential operator $(I-\Delta)^{\gamma/2}$ applied to $u \in \mathcal{D}'(\mathbb{T}^d)$ by

$$(I-\Delta)^{\frac{\gamma}{2}} u(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{\frac{\gamma}{2}} \, \hat{u}(k) \, e^{-2i\pi\langle k,x\rangle},$$

and denote

$$\|u\|_{\gamma,p} = \left\| (I-\Delta)^{\frac{\gamma}{2}} u \right\|_{L^p(\mathbb{T}^d)}. \tag{3}$$

As in [?, p.168], $H_p^\gamma(\mathbb{T}^d)$ is defined for $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ by

$$H_p^\gamma(\mathbb{T}^d) := \left\{ u \in \mathcal{D}'(\mathbb{T}^d); \, \|u\|_{\gamma,p} < \infty \right\}.$$

Bounded solution

Proposition Let $p > d$, $\lambda \in (0, 1)$, $u_0 \in H_p^\lambda$ and we assume that $u \in L^\infty([0, T] \times \mathbb{R}^d)$. Then the week solution of (1) satisfies $u \in C([0, T], H_p^\lambda)$.

Hypothesis and Results.

Let $\vartheta(x,v) = \vartheta^1(x)\vartheta^2(v)$ where ϑ^1 and ϑ^2 are probability densities on $\mathbb{T}^d$ and $\mathbb{R}^d$ such that

$$|\nabla_x \vartheta^1| \lesssim \vartheta^1 \tag{4}$$

and

$$\text{supp}(\vartheta^2) \subset B(0,1) \quad \text{and} \quad \int_{\mathbb{R}^2} \vartheta^2(v) v \mathrm{d}v = 0. \tag{5}$$

Let

$$\vartheta^N(x,v) = \vartheta^{1,N}(x)\vartheta^{2,N}(v) = N^{d\beta} N^{d\alpha} \vartheta(N^\beta x, N^\alpha v).$$

Hypothesis and Results.

We denote the empirical measure by

$$S_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_t^{i,N}, V_t^{i,N})}$$

and the regularized empirical measure by

$$F_t^N(x,v) = (S_t^N * \vartheta^N)(x,v). \tag{6}$$

Identity for F_t^N.

$$
\begin{aligned}
F_t^N(x,v) = F_0^N(x,v) &- \int_0^t \mathrm{div}_x \left(\vartheta^N * (\mathbf{v} S_s^N)\right)(x,v)\mathrm{d}s \\
&- \int_0^t \mathrm{div}_v \left(\vartheta^N * \left((\boldsymbol{\chi}_A(u_s^N) - \mathbf{v}) S_s^N\right)\right)(x,v)\mathrm{d}s \\
&+ M_t^N(x,v) + \frac{\sigma^2}{2}\int_0^t \Delta_v F_s^N(x,v)\mathrm{d}s.
\end{aligned}
\tag{7}
$$

Hypothesis and Results.

Assumption

- We assume that there exist three parameters $\alpha, \beta, \lambda \in [0,1]$, such that
- $d\beta + (d+1)\alpha < 1/2$
- $\alpha > \beta$;
- There exists $p > d$, $\lambda > \frac{d}{p}$ such that for any $q \geq 1$,

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left\|u_0^N\right\|_{\lambda,p}^q\right] < \infty \; ; \qquad (8)$$

- There exists some integer $k \geq 4$ such that for any $q \geq 1$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left\|\langle v\rangle^{2k}(\vartheta^N * S_0^N)\right\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}^q\right] < \infty \; ; \qquad (9)$$

  where $\langle v\rangle = (1+|v|^2)^{1/2}$.
- There exists some integer $k \geq 4$ such that $\int_{\mathbb{R}^d}\langle v\rangle^{2k}|F_0|^2 dv \in L^2(\mathbb{T}^d)$.

Theorem

We assume that the cut-off parameter $A \geq \|u\|_{L^\infty([0,T]\times\mathbb{T}^d)}$. Then we have

$$
\begin{aligned}
&(\mathbb{E}(\sup_{[0,T]} \|u^N - u\|_{\lambda,p})^q)^{1/q} \\
&\leq C_T(\mathbb{E}(\sup_{[0,t]} \|u_0^N - u_0\|_{\lambda,p})^q)^{1/q} \\
&+C_T(\mathbb{E}(\sup_{[0,t]} \|u_0^N - u_0\|_2)^q)^{1/2q} \\
&+C_T\Big(\mathbb{E}\big(\|\langle v\rangle^k(F_0^N - F_0 * \vartheta^N)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}\big)^q\Big)^{1/2q} \\
&+C_T\|\langle v\rangle^k(F_0 - F_0 * \vartheta^N)\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} + \sup_{t\in[0,T]} \left\|\langle v\rangle^k(\tilde{F}_t^N - F_t)\right\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} \\
&+C_T\left(\frac{1}{N^{\beta(\lambda-d/p)}} + \frac{1}{N^{\alpha-\beta}} + \frac{1}{N^{1/2-d\beta-(d+1)\alpha}}\right)
\end{aligned}
$$

Result.

Corollary

and that the cut-off parameter $A = \infty$. Then for any $\eta > 0$ and any $q \geq 1$, there exists $C > 0$ such that, for any $N \in \mathbb{N}^*$,

$$\mathbb{P}\left(\sup_{[0,T]} \|u^N - u\|_{\lambda,p} + \sup_{[0,T]} \|\langle v\rangle^k \big(F_t^N - F_t\big)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} \geq \eta\right)^{1/q}$$

$$\leq \frac{C}{\eta}(\mathbb{E}(\|u_0^N - u_0\|_2^2)^q)^{1/2q} + \frac{C}{\eta}(\mathbb{E}(\|u_0^N - u_0\|_{\lambda,p})^q)^{1/q}$$

$$+\frac{C_T}{\eta}\Big(\mathbb{E}\Big(\|\langle v\rangle^k (F_0^N - F * \vartheta^N)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}\Big)^q\Big)^{1/q}$$

$$+\frac{C_T}{\eta}\|\langle v\rangle^k (F_0 - F*\vartheta^N)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} + \frac{1}{\eta}\sup_{t\in[0,T]} \Big\|\langle v\rangle^k(\tilde{F}_t^N - F_t)\Big\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}$$

$$+\frac{C_T}{\eta}\left(\frac{1}{N^{\beta(\lambda-d/p)}} + \frac{1}{N^{\alpha-\beta}} + \frac{1}{N^{1/2-d\beta-(d+1)\alpha}}\right)$$

Result.

Theorem We assume that the cut-off parameter $A \geq \|u\|_{L^\infty([0,T]\times\mathbb{T}^d)}$ and $\int_{\mathbb{R}^d} \langle v\rangle^{2k} |F_t|^2 \mathrm{d}v \in L^\infty([0,T]\times\mathbb{T}^d)$. Then, it holds

$$
\Big(\mathbb{E}(\sup_{[0,T]} \|u^N - u\|_2^2)^q\Big)^{1/q} + \frac{1}{2}\Big(\mathbb{E}(\int_0^T \|\nabla(u_s^N - u_s)\|_2^2 ds)^q\Big)^{1/q}
$$

$$
+\Big(\mathbb{E}\big(\sup_{[0,T]} \|\langle v\rangle^k (F_s^N - F_s)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}\big)^q\Big)^{1/q}
$$

$$
\leq \Big(\mathbb{E}\|u_0^N - u_0\|_2^{2q}\Big)^{1/q}
$$

$$
+C_T\Big(\mathbb{E}\big(\|\langle v\rangle^k (F_0^N - F_0 * \vartheta^N)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}\big)^q\Big)^{1/q}
$$

$$
+C_T\|\langle v\rangle^k (F_0 - F_0 * \vartheta^N)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} + + \sup_{t\in[0,T]} \big\|\langle v\rangle^k (\tilde{F}_t^N - F_t)\big\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}
$$

$$
+C_T\left(\frac{1}{N^{2\beta(\lambda-d/p)}} + \frac{1}{+}\frac{1}{N^{2\alpha-2\beta}} + \frac{1}{N^{1-2d\beta-2(d+1)\alpha}}\right).
$$

Ideas of the proof .

$$
\begin{aligned}
F_t^N(x,v) = F_0^N(x,v) &- \int_0^t \mathrm{div}_x \left(\vartheta^N * (\mathbf{v} S_s^N)\right)(x,v)\mathrm{d}s \\
&- \int_0^t \mathrm{div}_v \left(\vartheta^N * \left((\boldsymbol{\chi}_A(u_s^N) - \mathbf{v}) S_s^N\right)\right)(x,v)\mathrm{d}s \\
&+ M_t^N(x,v) + \frac{\sigma^2}{2}\int_0^t \Delta_v F_s^N(x,v)\mathrm{d}s.
\end{aligned} \tag{10}
$$

Ideas of the proof .

There exists $C = C(A, k)$ such that

$$\sup_{N\in\mathbb{N}^*}\left(\mathbb{E}\sup_{t\in[0,T]}\|\langle v\rangle^k F_t^N\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}^{2q}\right.$$
$$\left.+\mathbb{E}\left[\left(\int_0^T\|\langle v\rangle^k\nabla_v F_s^N\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}^2\mathrm{d}s\right)^q\right]\right)^{1/q}<\infty. \tag{11}$$

For the proof we use Ito formula, commutator estimates, some functional inequality, the special properties of the mollifier $|\nabla_x\vartheta^1|\lesssim\vartheta^1$, The Burkholder-Davis-Gundy Inequality, etc.

Ideas of the proof .

The mild formulation then reads

$$
\begin{aligned}
u_t^N = e^{t\Delta} u_0^N - \int_0^t e^{(t-s)\Delta} P\big[(u_s^N \cdot \nabla)\boldsymbol{\chi}_A(u_s^N)\big]\mathrm{d}s \\
- \int_0^t e^{(t-s)\Delta} P\Big[\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\chi}_A(u_s^N(X_s^{i,N})) - V_s^{i,N})\,\delta^N_{X_s^{i,N}}\Big]\mathrm{d}s,
\end{aligned}
\tag{12}
$$

where P is the Leray projector.

Ideas of the proof .

Let $q \geq 1$, and choose $\gamma \in [0,1)$, $p \geq 2$ such that $\frac{\gamma}{2} + \frac{d}{2}(\frac{1}{2} - \frac{1}{p}) < 1$. Let the same assumptions as in Proposition **??** hold, and assume further that $\sup_{N \in \mathbb{N}^*} \mathbb{E}\|u_0^N\|_{\gamma,p}^q < \infty$. Then

$$\sup_{N \in \mathbb{N}^*} \mathbb{E}\Big[\sup_{t \in [0,T]} \left\| u_t^N \right\|_{\gamma,p}^q \Big] < \infty.$$

Ideas of the proof .

The aim is to estimate $\|F_t^N - F_t\|$ in some norm by decomposing it into $\|F_t^N - \tilde{F}_t^N\|$ and $\|\tilde{F}_t^N - F_t\|$, where

$$\tilde{F}_t^N = F_t * \vartheta^N. \tag{13}$$

We know that $\langle v\rangle^k F \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$, hence by a standard argument, the quantity

$$\varrho_N = \sup_{t\in[0,T]} \left\|\langle v\rangle^k (\tilde{F}_t^N - F_t)\right\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} \tag{14}$$

converges to 0.

Ideas of the proof .

For any $t \leq T$ and any $N \in \mathbb{N}^*$, we have

$$\Bigg(\mathbb{E} \sup_{s\in[0,t]} \|\langle v\rangle^k (F_s^N - \tilde{F}_s^N)\|^{2q}_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} \tag{15}$$

$$+ \mathbb{E}\Bigg[\Big(\int_0^t \|\langle v\rangle^k \nabla_v (F_s^N - \tilde{F}_s^N)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} \mathrm{d}s\Big)^q\Bigg]\Bigg)^{1/q}$$

$$\leq C\big(\mathbb{E}\|\langle v\rangle^k (F_0^N - \tilde{F}_0^N)\|^{2q}_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}\big)^{1/q} \tag{16}$$

$$+ C\Big(\frac{1}{N^{2\beta(\gamma-d/p)}} + \frac{1}{N^{2\alpha-2\beta}} + \frac{1}{N^{1-2d\beta-2(d+1)\alpha}}\Big)$$

$$+ 4\int_0^t \big(\mathbb{E} \sup_{r\in[0,s]} \|\langle v\rangle^k (F_r^N - \tilde{F}_r^N)\|^{2q}_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}\big)^{1/q} \mathrm{d}s \tag{17}$$

$$+ 4\int_0^t \big(\mathbb{E}\|u_s - u_s^N\|^{2q}_{L^m(\mathbb{T}^d)}\big)^{1/q} \mathrm{d}s,$$

with $m = \infty$.Furthermore, if

$$\int_{\mathbb{R}^d} \langle v\rangle^{2k} |F_t|^2 \mathrm{d}v \in L^\infty([0,T]\times\mathbb{T}^d),$$

then (15) holds with $m = 2$.

Ideas of the proof .

$$u_t^N - u_t = e^{t\Delta}(u_0^N - u_0) - \int_0^t e^{(t-s)\Delta} P\nabla(u_s^N \boldsymbol{\chi}_A(u_s^N) - u_s \boldsymbol{\chi}_A(u_s))ds$$

$$-\int_0^t e^{(t-s)\Delta} P[\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\chi}_A(u_s^N)(X_s^{i,N}) - V_s^{i,N})\delta^N_{X_s^{i,N}} - (u_s m_0(F_s) - m_1(F_s))]ds.$$

Ideas of the proof .

$$
\begin{aligned}
&\left(\mathbb{E}\|u_t^N - u_t\|_{\gamma,p}^q\right)^{\frac{1}{q}} \\
&\qquad \lesssim \Bigg(\left(\mathbb{E}\|u_0^N - u_0\|_{\gamma,p}^q\right)^{\frac{1}{q}} + C_T\big(\mathbb{E}\|\langle v\rangle^k (F_0^N - \vartheta^N * F_0)\|_{L^2(\mathbb{T}^d}^q \\
&\qquad\quad + C\Big(\frac{1}{N^{\beta(\gamma-d/p)}} + \frac{1}{N^{\alpha-\beta}} + \frac{1}{N^{1/2-d\beta-(d+))\alpha}}\Big) + \varrho_N \\
&\qquad\quad + 4\int_0^t \Big(\mathbb{E} \sup_{r\in[0,s]} \|\langle v\rangle^k (F_r^N - \tilde{F}_r^N)\|_{L^2(\mathbb{T}^d\times\mathbb{R}^d)}^q\Big)^{\frac{1}{q}} \mathrm{d}s \Bigg).
\end{aligned}
$$

Future works .

- Find some setting to be able to cover the week solution class.
- Consider the case compressible Euler/Navier Stoke-Vlasov systems from particles models. Maybe follow
  K Oelschläger, On the connection between Hamiltonian many-particle systems and the hydrodynamical equations, Archive for rational mechanics and analysis, 1991.

Microscopic model.

- We introduce the mollified delta Dirac function

$$\delta^N_{X^{i,N}_t}(x) = \vartheta^{1,N}(x - X^{i,N}_t).$$

$$\begin{cases} \partial_t u^N_t(x) - \Delta u^N_t(x) + (u^N_t \cdot \nabla)\left[\boldsymbol{\chi}_A(u^N_t)\right](x) + \nabla p^N_t(x) \\ = -\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\chi}_A(u^N_t(X^{i,N}_t)) - V^{i,N}_t)\delta^N_{X^{i,N}_t}(x), \\ \operatorname{div} u^N_t = 0 \\ u^N(0,x) = u^N_0(x), \quad x \in \mathbb{T}^d, \\ dX^{i,N}_t = V^{i,N}_t \mathrm{d}t \\ dV^{i,N}_t = (\boldsymbol{\chi}_A(u^N_t(X^{i,N}_t)) - V^{i,N}_t)\mathrm{d}t + \sigma \mathrm{d}B^i_t + dW \end{cases} \tag{18}$$

harmonise $u(t,\cdot)$ and $u_t(\cdot)$ where N is the number of particles and $(X^{i,N}_t, V^{i,N}_t)$ for $1 \leq i \ \leq N$ are position and velocity of the i^{th} particle.

Bibliography.

- K. Oelschläger, A law of large numbers for moderately interacting diffusion processes, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 1985.
- B. Jourdain and S. Méléard, Propagation of chaos and fluctuations for a moderate model with smooth initial data, Ann. Inst. H. Poincaré Probab. Statist, 1998.
- F. Flandoli and M. Leocata, *A particle system approach to aggregation phenomena*, J. Appl. Probab., 56, pp. 282–306, 2019.
- F. Flandoli, C. Olivera, and M. Simon, Uniform approximation of 2 dimensional Navier-Stokes equation by stochastic interacting particle systems, SIAM J. Math. Anal, 2020.
- Flandoli, F., Leocata, M. Ricci, C. The Navier-Stokes-Vlasov-Fokker-Planck System as a Scaling Limit of Particles in a Fluid. J. Math. Fluid Mech., 2021.

Bibliography.

- C. Olivera, A. Richard, and M. Tomašević, Quantitative particle approximation of nonlinear Fokker-Planck equations with singular kerne, Ann. Sc. Norm. Super. Pisa Cl. Sci, 2021.
- L. Goudenège, C. Olivera, G. Planas, A. Richard, Quantitative approximation of the Navier-Stokes-Vlasov-Fokker-Planck System by stochastic particles systems, in preparation.