Singular SPDEs with the Cauchy-Riemann operator on a torus

Evelina Shamarova

Federal University of Paraíba

Joint work with Zdzisław Brzeźniak and Mikhail Neklyudov

Cauchy-Riemann SPDE on the torus

Consider a $\mathbb{C}^3$-valued equation on a 2D torus $\mathbb{T}^2$:

$$\partial_{\bar{z}} r = r \times \bar{r} + i\,\gamma\,\mathscr{W}, \qquad \text{(CR)}$$

where $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$ is the Cauchy-Riemann operator on $\mathbb{T}^2$, $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3)$ is a real 3D white noise on $\mathbb{T}^2$ whose component $\mathscr{W}_3$ has zero mean over $\mathbb{T}^2$, $\gamma := (\gamma_1, \gamma_2, \gamma_3)$ is an $\mathbb{R}^3$-vector and $\gamma\,\mathscr{W} := (\gamma_1\mathscr{W}_1, \gamma_2\mathscr{W}_2, \gamma_3\mathscr{W}_3)$.

Cauchy-Riemann SPDE on the torus

Consider a $\mathbb{C}^3$-valued equation on a 2D torus $\mathbb{T}^2$:

$$\partial_{\bar{z}} r = r \times \bar{r} + i\,\gamma\,\mathscr{W}, \tag{CR}$$

where $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$ is the Cauchy-Riemann operator on $\mathbb{T}^2$, $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3)$ is a real 3D white noise on $\mathbb{T}^2$ whose component $\mathscr{W}_3$ has zero mean over $\mathbb{T}^2$, $\gamma := (\gamma_1, \gamma_2, \gamma_3)$ is an $\mathbb{R}^3$-vector and $\gamma\,\mathscr{W} := (\gamma_1\mathscr{W}_1, \gamma_2\mathscr{W}_2, \gamma_3\mathscr{W}_3)$.

- The Cauchy-Riemann operator $\partial_{\bar{z}}$ and its complex conjugate $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ satisfy the identity

$$\partial_z \partial_{\bar{z}} = \frac{1}{4}\Delta$$

so the operator $(\partial_{\bar{z}})^{-1}$ increases the regularity by 1.

Motivation

- Cauchy-Riemann operator $\partial_{\bar{z}}$ and its complex conjugate ∂_z are building blocks of the 2D free Dirac operator

$$\mathcal{D} := -2i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}.$$

Motivation

- Cauchy-Riemann operator $\partial_{\bar{z}}$ and its complex conjugate ∂_z are building blocks of the 2D free Dirac operator

$$\mathcal{D} := -2i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}.$$

- Equation (CR) can be written in the following equivalent form

$$\mathcal{D} \begin{pmatrix} \bar{r} \\ r \end{pmatrix} = \begin{pmatrix} 2i \\ -2i \end{pmatrix} r \times \bar{r} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} (\gamma \mathscr{W}).$$

Motivation

- Cauchy-Riemann operator $\partial_{\bar{z}}$ and its complex conjugate ∂_z are building blocks of the 2D free Dirac operator

$$\mathcal{D} := -2i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}.$$

- Equation (CR) can be written in the following equivalent form

$$\mathcal{D} \begin{pmatrix} \bar{r} \\ r \end{pmatrix} = \begin{pmatrix} 2i \\ -2i \end{pmatrix} r \times \bar{r} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} (\gamma \mathscr{W}).$$

- Dirac operators have applications in quantum mechanics, relativistic field theory, differential geometry, and Clifford algebras.

Motivation

- The quadratic term $r \times \bar{r}$ represents a nonlinear self-interaction, introduces an interesting mathematical structure.

Motivation

- The quadratic term $r \times \bar{r}$ represents a nonlinear self-interaction, introduces an interesting mathematical structure.

- One possible interpretation of this cross-product term is that it describes a nonlinear coupling between spinor-like components in a quantum mechanical system.

Motivation

- The quadratic term $r \times \bar{r}$ represents a nonlinear self-interaction, introduces an interesting mathematical structure.
- One possible interpretation of this cross-product term is that it describes a nonlinear coupling between spinor-like components in a quantum mechanical system.
- Nonlinearities of quadratic type appear, for example, in Dirac-Klein-Gordon systems.

Main result

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3)$ *be a 3D white noise on* $\mathbb{T}^2$ *such that* $\mathscr{W}_3$ *has zero mean over* $\mathbb{T}^2$*. Further let* $\mathscr{W}^\varepsilon$ *be a mollification of* $\mathscr{W}$ *by the standard mollifier. Assume that* $\kappa \in (0, \frac{1}{2})$*. Then, for every* $\delta > 0$*, there exists a number* $\varsigma > 0$ *and a set* $\Omega_\delta \subset \Omega$*, depending also on* κ*, with* $\mathbb{P}(\Omega_\delta) > 1 - \delta$*, such that for every* $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$ *with* $|\gamma| < \varsigma$*, and for every* $\varepsilon > 0$*, the system*

$$\partial_{\bar{z}} r^\varepsilon = r^\varepsilon \times \overline{r^\varepsilon} + i\,\gamma\,\mathscr{W}^\varepsilon, \qquad \text{(mCR)}$$

considered for $\omega \in \Omega_\delta$*, has a solution* $r^\varepsilon \in \mathrm{C}^\infty(\mathbb{T}^2, \mathbb{C}^3)$ *such that ...*

Main rerult (continued)

Theorem

... there exists a limit

$$r := \lim_{\varepsilon \to 0} r^{\varepsilon} \quad \textit{in} \quad \mathrm{C}^{-\kappa}(\mathbb{T}^2) \quad \textit{for every } \omega \in \Omega_\delta.$$

Moreover, the convergence holds with the following rate:

$$\|r^{\varepsilon} - r\|_{\mathrm{C}^{-\kappa}(\mathbb{T}^2)} \lesssim \varepsilon^{\frac{\kappa}{2}} \quad \textit{uniformly in} \quad \omega \in \Omega_\delta.$$

Periodic Hölder-Zygmund space

- Our proof uses a *fixed-point argument* in a Hölder-Zygmund space.

Periodic Hölder-Zygmund space

- Our proof uses a *fixed-point argument* in a Hölder-Zygmund space.
- Our definition of the latter space is adapted to the periodic setting.

Periodic Hölder-Zygmund space

- Our proof uses a *fixed-point argument* in a Hölder-Zygmund space.
- Our definition of the latter space is adapted to the periodic setting.
- We start by introducing the dyadic interval

$$[0,1]_{\mathbb{D}} := \{2^{-n},\ n \in \mathbb{N}_0\}.$$

Periodic Hölder-Zygmund space

- Our proof uses a *fixed-point argument* in a Hölder-Zygmund space.
- Our definition of the latter space is adapted to the periodic setting.
- We start by introducing the dyadic interval
$$[0,1]_{\mathbb{D}} := \{2^{-n},\ n \in \mathbb{N}_0\}.$$
- Let $\varphi \in \mathcal{D}(\mathbb{T}^d)$, define the rescaled centered function
$$\varphi_x^\lambda(y) := \frac{1}{\lambda^d}\varphi\Big(\frac{y-x}{\lambda}\Big), \qquad \lambda \in [0,1]_{\mathbb{D}},\ x \in \mathbb{T}^d.$$

Periodic Hölder-Zygmund space

- Our proof uses a *fixed-point argument* in a Hölder-Zygmund space.
- Our definition of the latter space is adapted to the periodic setting.
- We start by introducing the dyadic interval
$$[0,1]_{\mathbb{D}} := \{2^{-n},\ n \in \mathbb{N}_0\}.$$
- Let $\varphi \in \mathcal{D}(\mathbb{T}^d)$, define the rescaled centered function
$$\varphi_x^\lambda(y) := \frac{1}{\lambda^d}\varphi\Big(\frac{y-x}{\lambda}\Big), \qquad \lambda \in [0,1]_{\mathbb{D}},\ x \in \mathbb{T}^d.$$
- Note that $\varphi_x^\lambda \in \mathcal{D}(\mathbb{T}^d)$.

Periodic Hölder-Zygmund space

Let $r \in \mathbb{N}_0$ and $\alpha \geqslant 0$. Define the classes of test functions

$$\begin{cases} \mathscr{B}^r := \{\varphi \in \mathcal{D}(\mathbb{T}^d) : \|\partial^k \varphi\|_\infty \leqslant 1, \text{ for all } 0 \leqslant |k| \leqslant r\}, \\ \mathscr{B}_\alpha := \{\varphi \in \mathcal{D}(\mathbb{T}^d) : \int_{\mathbb{T}^d} \varphi(z) z^k dz = 0, \text{ for all } 0 \leqslant |k| \leqslant \alpha\}, \\ \mathscr{B}^r_\alpha := \mathscr{B}^r \cap \mathscr{B}_\alpha, \end{cases}$$

where $k = (k_1, \ldots, k_d)$; and $z^k := z_1^{k_1} \ldots z_d^{k_d}$, $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$.

Definition (Periodic Hölder-Zygmund spaces $\mathscr{Z}^{\alpha}(\mathbb{T}^d)$)

Let $\alpha \in \mathbb{R}$*. Define a "pseudo-norm"* $|f|_{\mathscr{Z}^\alpha}$ *for* $f \in \mathcal{D}'(\mathbb{T}^d)$ *as follows.*

If $\alpha \geqslant 0$*, then we put*

$$|f|_{\mathscr{Z}^\alpha} := \sup_{\psi \in \mathscr{B}^0} |f(\psi)| + \sup_{\substack{x\in\mathbb{T}^d,\, \lambda\in[0,1]_{\mathbb{D}},\\ \varphi\in\mathscr{B}^0_\alpha}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\alpha}.$$

If $\alpha < 0$*, then we put*

$$|f|_{\mathscr{Z}^\alpha} := \sup_{\substack{x\in\mathbb{T}^d,\, \lambda\in[0,1]_{\mathbb{D}},\\ \varphi\in\mathscr{B}^{[-\alpha+1]}}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\alpha}.$$

We define the periodic Hölder-Zygmund space $\mathscr{Z}^\alpha := \mathscr{Z}^\alpha(\mathbb{T}^d)$ *as the space of distributions* $f \in \mathcal{D}'(\mathbb{T}^d)$ *for which* $|f|_{\mathscr{Z}^\alpha} < \infty$*.*

Remark

Our norm resembles that in

L. Broux, F. Caravenna and L. Zambotti, Hairer's multilevel Schauder estimates without regularity structures, Trans. Am. Math. Soc. 2024.

However, we define Hölder-Zygmund spaces using

periodic test functions and dyadic scales.

If $\alpha \notin \mathbb{N}$, the definitions are equivalent, as we show in our paper.

For $\alpha \in \mathbb{N}$, $\mathscr{Z}^\alpha(\mathbb{T}^d)$ defines a distinct space, which is not utilized in our analysis.

Periodic little Hölder-Zygmund spaces

Remark

It is known that classical Hölder-Zygmund spaces are not separable and that the space C^∞ is not dense in them. For this reason, we define so called little Hölder-Zygmund spaces $\mathscr{Z}^\alpha$.

Periodic little Hölder-Zygmund spaces

Remark

It is known that classical Hölder-Zygmund spaces are not separable and that the space C^∞ is not dense in them. For this reason, we define so called little Hölder-Zygmund spaces $\mathscr{z}^\alpha$.

Definition (Periodic little Hölder-Zygmund spaces)

We define a periodic little Hölder-Zygmund space

$$\mathscr{z}^\alpha := \mathscr{z}^\alpha(\mathbb{T}^d) \quad \text{as the closure of } \mathrm{C}^\infty(\mathbb{T}^d) \text{ in } \mathscr{Z}^\alpha(\mathbb{T}^d).$$

We furthermore denote by $|\cdot|_\alpha$, $\alpha \in \mathbb{R}$, *the restriction of the norm* $|\cdot|_{\mathscr{Z}^\alpha}$ *to* $\mathscr{z}^\alpha(\mathbb{T}^d)$.

Periodic little Hölder-Zygmund spaces

Remark

A Gaussian white noise $\mathscr{W}$ on $\mathbb{T}^d$, as well as its convolutions $K * \mathscr{W}$ and $G * \mathscr{W}$ are elements of little Hölder-Zygmund spaces. Furthermore, Young products are well defined (specifically, uniquely defined) when they are restricted to little Hölder-Zygmund spaces.

Periodic little Hölder-Zygmund spaces

Remark

A Gaussian white noise $\mathscr{W}$ on $\mathbb{T}^d$, as well as its convolutions $K * \mathscr{W}$ and $G * \mathscr{W}$ are elements of little Hölder-Zygmund spaces. Furthermore, Young products are well defined (specifically, uniquely defined) when they are restricted to little Hölder-Zygmund spaces.

Lemma

Let $\mathscr{W} : \Omega \to \mathcal{D}'(\mathbb{T}^d)$ be a Gaussian white noise. Then, for any $\kappa > 0$, $\mathscr{W} \in \mathscr{Z}^{-\frac{d}{2}-\kappa}(\mathbb{T}^d)$ a.s. Furthermore, for any $\varepsilon > 0$ and $\kappa > \varkappa > 0$,

$$\left|\mathscr{W}^{\varepsilon} - \mathscr{W}\right|_{-\frac{d}{2}-\kappa} \lesssim \varepsilon^{\kappa-\varkappa}\left|\mathscr{W}\right|_{-\frac{d}{2}-\varkappa}, \quad \text{a.s.}$$

Green function for the Cauchy-Riemann operator

Let K be the Green function for the Laplacian on $\mathbb{T}^2$. We define

$$G := 2\partial_z K = \partial_1 K - i\partial_2 K.$$

The fixed-point argument

$$\begin{cases} \partial_{\bar z} r_1^\varepsilon = r_2^\varepsilon \overline{r_3^\varepsilon} - \overline{r_2^\varepsilon} r_3^\varepsilon + \gamma_1 i \mathscr{W}_1^\varepsilon, \\ \partial_{\bar z} r_2^\varepsilon = \overline{r_1^\varepsilon} r_3^\varepsilon - r_1^\varepsilon \overline{r_3^\varepsilon} + \gamma_2 i \mathscr{W}_2^\varepsilon, \\ \partial_{\bar z} r_3^\varepsilon = r_1^\varepsilon \overline{r_2^\varepsilon} - \overline{r_1^\varepsilon} r_2^\varepsilon + \gamma_3 i \mathscr{W}_3^\varepsilon, \end{cases} \qquad \text{(mCR)}$$

The fixed-point argument

$$\begin{cases} \partial_{\bar z} r_1^\varepsilon = r_2^\varepsilon \overline{r_3^\varepsilon} - \overline{r_2^\varepsilon} r_3^\varepsilon + \gamma_1 i \mathscr{W}_1{}^\varepsilon, \\ \partial_{\bar z} r_2^\varepsilon = \overline{r_1^\varepsilon} r_3^\varepsilon - r_1^\varepsilon \overline{r_3^\varepsilon} + \gamma_2 i \mathscr{W}_2{}^\varepsilon, \\ \partial_{\bar z} r_3^\varepsilon = r_1^\varepsilon \overline{r_2^\varepsilon} - \overline{r_1^\varepsilon} r_2^\varepsilon + \gamma_3 i \mathscr{W}_3{}^\varepsilon, \end{cases} \tag{mCR}$$

Furthermore, introduce

$$\xi_k^\varepsilon := 2iG * \mathscr{W}_k^\varepsilon = 2G_2 * \mathscr{W}_k^\varepsilon + 2G_1 * \mathscr{W}_k^\varepsilon, \qquad k = 1, 2, 3,$$

$$G := G_1 + iG_2.$$

Lemma

If $f \in \mathrm{C}^{\alpha}(\mathbb{T}^2)$, $\alpha \in (0,1)$, $C \in \mathbb{C}$ *and a function* $r : \mathbb{T}^2 \to \mathbb{C}$ *satisfies*

$$r = G * f + C$$

then $r \in \mathrm{C}^{1+\alpha}(\mathbb{T}^2)$ *and satisfies the identity*

$$-2\partial_{\bar{z}} r = f - [f],$$

where $[f] := (4\pi^2)^{-1} \int f$ *denotes the average over* $\mathbb{T}^2$*.*

The fixed-point argument

Proposition

Let $c > 0$, $\varepsilon > 0$ and $\gamma_k \in \mathbb{R}$, $k = 1, 2, 3$. There exist $\mathbb{C}$-valued Gaussian mean zero random variables a and b such that if $(r_1^\varepsilon, r_2^\varepsilon, r_3^\varepsilon)$ is a $\mathrm{C}^\alpha(\mathbb{T}^2)$-pathwise solution of the system (for some $\alpha \in (0,1)$),

$$\begin{cases} r_1^\varepsilon = 2G * \left(\overline{r_2^\varepsilon} r_3^\varepsilon - r_2^\varepsilon \overline{r_3^\varepsilon}\right) - \gamma_1 \xi_1^\varepsilon + a, \\ r_2^\varepsilon = 2G * \left(r_1^\varepsilon \overline{r_3^\varepsilon} - \overline{r_1^\varepsilon} r_3^\varepsilon\right) - \gamma_2 \xi_2^\varepsilon + b, \\ r_3^\varepsilon = 2G * \left(\overline{r_1^\varepsilon} r_2^\varepsilon - r_1^\varepsilon \overline{r_2^\varepsilon}\right) - \gamma_3 \xi_3^\varepsilon + c, \end{cases}$$

then $r_k^\varepsilon \in \mathrm{C}^\infty(\mathbb{T}^2)$, $k = 1, 2, 3$, and $(r_1^\varepsilon, r_2^\varepsilon, r_3^\varepsilon)$ solves (mCR).

The fixed-point argument

Define a and b by the formulas

$$2ac = i\gamma_2\eta_2, \qquad 2bc = -i\gamma_1\eta_1,$$

where η_1 and η_2 are independent standard real Gaussian random variables which are zero Fourier coefficients for $\mathscr{W}_1$ and $\mathscr{W}_2$.

The fixed-point argument

Lemma

For $\varepsilon > 0$, let

$$R^\varepsilon := r^\varepsilon + \gamma\xi^\varepsilon,$$

where $\gamma\xi^\varepsilon := (\gamma_1\xi_1^\varepsilon, \gamma_2\xi_2^\varepsilon, \gamma_3\xi_3^\varepsilon)$. Then, the above system is equivalent to

$$R^\varepsilon = -2G * \left(R^\varepsilon \times \overline{R^\varepsilon} - (\gamma\xi^\varepsilon) \times \overline{R}^\varepsilon - R^\varepsilon \times (\gamma\overline{\xi^\varepsilon})\right) - \tilde{\gamma}\zeta^\varepsilon + \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where

$$\zeta^\varepsilon := 2G * (\xi^\varepsilon \times \overline{\xi^\varepsilon}), \quad \tilde{\gamma} := (\gamma_2\gamma_3, \gamma_1\gamma_3, \gamma_1\gamma_2), \quad \tilde{\gamma}\zeta^\varepsilon := (\tilde{\gamma}_1\zeta_1^\varepsilon, \tilde{\gamma}_2\zeta_2^\varepsilon, \tilde{\gamma}_3\zeta_3^\varepsilon).$$

The fixed-point argument

- For every $\nu \in \mathbb{R}$, introduce the Banach space

$$E_\nu = \big\{ R = (R_1, R_2, R_3) : \mathbb{T}^2 \to \mathbb{C}^3 : \quad R_k \in \mathscr{X}^\nu(\mathbb{T}^2), k = 1, 2, 3 \big\}$$

The fixed-point argument

- For every $\nu \in \mathbb{R}$, introduce the Banach space

$$E_\nu = \{R = (R_1, R_2, R_3) : \mathbb{T}^2 \to \mathbb{C}^3 : \quad R_k \in \mathscr{X}^\nu(\mathbb{T}^2), k = 1, 2, 3\}$$

- For every $\sigma > 0$, we set $c = \frac{\sigma}{4}$ and define a, b via c.

The fixed-point argument

- For every $\nu \in \mathbb{R}$, introduce the Banach space

$$E_\nu = \left\{ R = (R_1, R_2, R_3) : \mathbb{T}^2 \to \mathbb{C}^3 : \quad R_k \in \mathscr{X}^\nu(\mathbb{T}^2), k = 1, 2, 3 \right\}$$

- For every $\sigma > 0$, we set $c = \frac{\sigma}{4}$ and define a, b via c.
- Given $\varepsilon > 0$, $\sigma > 0$, and $\kappa \in (0, \frac{1}{2})$, define a map

$$\Gamma_{\varepsilon,\sigma} : E_{1-\kappa} \to E_{1-\kappa},$$
$$\Gamma_{\varepsilon,\sigma} R := -2G * \left(R \times \overline{R} + (\gamma \xi^\varepsilon) \times \overline{R} + R \times (\gamma \overline{\xi^\varepsilon}) \right) - \tilde{\gamma} \zeta^\varepsilon + (a, b, c)^\top.$$

The fixed-point argument

- For every $\nu \in \mathbb{R}$, introduce the Banach space

$$E_\nu = \{R = (R_1, R_2, R_3) : \mathbb{T}^2 \to \mathbb{C}^3 : \quad R_k \in \mathscr{X}^\nu(\mathbb{T}^2), k = 1, 2, 3\}$$

- For every $\sigma > 0$, we set $c = \frac{\sigma}{4}$ and define a, b via c.
- Given $\varepsilon > 0$, $\sigma > 0$, and $\kappa \in (0, \frac{1}{2})$, define a map

$$\Gamma_{\varepsilon,\sigma} : E_{1-\kappa} \to E_{1-\kappa},$$
$$\Gamma_{\varepsilon,\sigma} R := -2G * \Big(R \times \overline{R} + (\gamma \xi^\varepsilon) \times \overline{R} + R \times (\gamma \overline{\xi^\varepsilon})\Big) - \tilde{\gamma} \zeta^\varepsilon + (a, b, c)^\top.$$

- $\zeta^\varepsilon := 2G * (\xi^\varepsilon \times \overline{\xi^\varepsilon})$

A note on $\zeta^\varepsilon := 2G * (\xi^\varepsilon \times \overline{\xi^\varepsilon})$

Lemma

Let $\kappa \in (0, \frac{1}{2})$. Then ζ^ε has a limit ζ in $E_{1-\kappa}$ as $\varepsilon \to 0$. Furthermore, for all $\varepsilon \geqslant 0$ and $\kappa \in (0, \frac{1}{2})$,

$$\|\zeta^\varepsilon\|_{1-\kappa} \lesssim \|\mathscr{W}\|^2_{-1-\kappa},$$

where $\zeta^0 := \zeta$, and the constant in the above inequality does not depend on $\varepsilon \geqslant 0$.

A note on $\zeta^\varepsilon := 2G * (\xi^\varepsilon \times \overline{\xi^\varepsilon})$

Lemma

Let $\kappa \in (0, \frac{1}{2})$. Then ζ^ε has a limit ζ in $E_{1-\kappa}$ as $\varepsilon \to 0$. Furthermore, for all $\varepsilon \geqslant 0$ and $\kappa \in (0, \frac{1}{2})$,

$$\|\zeta^\varepsilon\|_{1-\kappa} \lesssim \|\mathscr{W}\|^2_{-1-\kappa},$$

where $\zeta^0 := \zeta$, and the constant in the above inequality does not depend on $\varepsilon \geqslant 0$.

The above lemma is a fundamental result of this work, underpinning the proof of the main result. Its proof is postponed to next slides.

The fixed-point argument

- Consider the closed ball $M_{1-\kappa,\sigma} := \{R \in E_{1-\kappa} : \|R\|_{1-\kappa} \leqslant \sigma\}$, of radius $\sigma > 0$ (to be fixed later), which is a complete metric space.

The fixed-point argument

- Consider the closed ball $M_{1-\kappa,\sigma} := \{R \in E_{1-\kappa} : \|R\|_{1-\kappa} \leqslant \sigma\}$, of radius $\sigma > 0$ (to be fixed later), which is a complete metric space.

Proposition

For every $\delta \in (0,1)$ and $\kappa \in (0, \frac{1}{2})$, there exists a set $\Omega_\delta \subset \Omega$, $\mathbb{P}(\Omega_\delta) > 1 - \delta$, and a number $\sigma > 0$ such that for all $\omega \in \Omega_\delta$, $\varepsilon \geqslant 0$, and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ satisfying $|\gamma| \lesssim \sigma^2$,

$$\Gamma_{\varepsilon,\sigma} : M_{1-\kappa,\sigma} \to M_{1-\kappa,\sigma}$$

is a strict contraction and hence has a unique fixed point in $M_{1-\kappa,\sigma}$.

Moreover, the contraction constant does not depend on $\varepsilon \geqslant 0$ and $\omega \in \Omega_\delta$.

The fixed-point argument

Remark

The proposition is also valid for the map $\Gamma_{0,\sigma}$. In particular, ξ^0 and ζ^0 are well-defined. The products involved in $\Gamma_{0,\sigma}$ are understood as Young products.

Outline of proof

- Let $\delta \in (0,1)$. $\exists\, \Lambda \in (0,+\infty)$ such that if

$$\Omega_\delta := \Big\{\omega : \|\mathscr{W}\|_{-1-\kappa} + |\eta_1| + |\eta_2| < \Lambda\Big\},$$

then $\mathbb{P}(\Omega_\delta) > 1-\delta$.

Outline of proof

- Let $\delta \in (0,1)$. $\exists\, \Lambda \in (0,+\infty)$ such that if

$$\Omega_\delta := \Big\{\omega : \|\mathscr{W}\|_{-1-\kappa} + |\eta_1| + |\eta_2| < \Lambda\Big\},$$

  then $\mathbb{P}(\Omega_\delta) > 1-\delta$.
- One can bound $\|\xi^\varepsilon\|_{-\kappa}$ and $\|\zeta^\varepsilon\|_{-\kappa}$ by $\|\mathscr{W}\|_{-1-\kappa}$ uniformly in $\varepsilon \geqslant 0$.

Outline of proof

- Let $\delta \in (0,1)$. $\exists\, \Lambda \in (0,+\infty)$ such that if
$$\Omega_\delta := \Big\{\omega : \|\mathscr{W}\|_{-1-\kappa} + |\eta_1| + |\eta_2| < \Lambda\Big\},$$
then $\mathbb{P}(\Omega_\delta) > 1 - \delta$.
- One can bound $\|\xi^\varepsilon\|_{-\kappa}$ and $\|\zeta^\varepsilon\|_{-\kappa}$ by $\|\mathscr{W}\|_{-1-\kappa}$ uniformly in $\varepsilon \geqslant 0$.
- If $c \sim \sigma$ and $|\gamma| \sim \sigma^2$, one can prove that
$$\|\Gamma_{\varepsilon,\sigma} R\|_{1-\kappa} \leqslant \sigma \quad \text{on} \;\; \Omega_\delta.$$

Outline of proof

- Let $\delta \in (0,1)$. $\exists\, \Lambda \in (0,+\infty)$ such that if

$$\Omega_\delta := \Big\{\omega : \|\mathscr{W}\|_{-1-\kappa} + |\eta_1| + |\eta_2| < \Lambda\Big\},$$

  then $\mathbb{P}(\Omega_\delta) > 1-\delta$.

- One can bound $\|\xi^\varepsilon\|_{-\kappa}$ and $\|\zeta^\varepsilon\|_{-\kappa}$ by $\|\mathscr{W}\|_{-1-\kappa}$ uniformly in $\varepsilon \geqslant 0$.
- If $c \sim \sigma$ and $|\gamma| \sim \sigma^2$, one can prove that

$$\|\Gamma_{\varepsilon,\sigma} R\|_{1-\kappa} \leqslant \sigma \quad \text{on} \;\; \Omega_\delta.$$

- and for $R, \hat{R} \in M_{1-\kappa,\sigma}$,

$$\|\Gamma_{\varepsilon,\sigma} R - \Gamma_{\varepsilon,\sigma}\hat{R}\|_{1-\kappa} \leqslant C\|R-\hat{R}\|_{1-\kappa} \quad C < 1.$$

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

Lemma

Let $A \in \mathbb{R}$ *be a constant and let for* $\alpha \in (0,1)$, $g_k \in \mathrm{C}^{\alpha}(\mathbb{T}^2, \mathbb{R})$ *and* $f_k = iG * g_k$, *for* $k = 1, 2$. *Then,*

$$f_1 \overline{f_2} - \overline{f_1} f_2 = \partial_1\big((K * g_1 - A)\,\partial_2 K * g_2\big) - \partial_2\big((K * g_1 - A)\,\partial_1 K * g_2\big).$$

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

Lemma

Let $A \in \mathbb{R}$ be a constant and let for $\alpha \in (0,1)$, $g_k \in \mathrm{C}^{\alpha}(\mathbb{T}^2, \mathbb{R})$ and $f_k = iG * g_k$, for $k = 1, 2$. Then,

$$f_1\overline{f_2} - \overline{f_1}f_2 = \partial_1\big((K * g_1 - A)\,\partial_2 K * g_2\big) - \partial_2\big((K * g_1 - A)\,\partial_1 K * g_2\big).$$

Idea of Proof: Since $G = \partial_{\bar{z}} K$, the result follows by differentiation of the products. The constant A cancels by differentiation.

Lemma

The components of $\vartheta := \xi \times \overline{\xi}$ *satisfy the following relations:*

$$\begin{cases} \vartheta_1^\varepsilon = 2\big(\partial_1[(K * \mathscr{W}_2{}^\varepsilon)\xi_{23}^\varepsilon] - \partial_2[(K * \mathscr{W}_2{}^\varepsilon)\xi_{13}^\varepsilon]\big), \\ \vartheta_2^\varepsilon = 2\big(\partial_1[(K * \mathscr{W}_3{}^\varepsilon)\xi_{21}^\varepsilon] - \partial_2[(K * \mathscr{W}_3{}^\varepsilon)\xi_{11}^\varepsilon]\big), \\ \vartheta_3^\varepsilon = 2\big(\partial_1[(K * \mathscr{W}_1{}^\varepsilon)\xi_{22}^\varepsilon] - \partial_2[(K * \mathscr{W}_1{}^\varepsilon)\xi_{12}^\varepsilon]\big), \end{cases}$$

where $\xi_{1k}^\varepsilon = \mathrm{Im}\,\xi_k$, $\xi_{2k}^\varepsilon = \mathrm{Re}\,\xi_k$.

Furthermore, ϑ^ε *satisfy, for any* $x \in \mathbb{T}^2$, *the additional relations*

$$\begin{cases} \vartheta_1^\varepsilon = 2\big(\partial_1[(K * \mathscr{W}_2{}^\varepsilon - K * \mathscr{W}_2{}^\varepsilon(x))\xi_{23}^\varepsilon] - \partial_2[(K * \mathscr{W}_2{}^\varepsilon - K * \mathscr{W}_2{}^\varepsilon(x))\xi_{13}^\varepsilon]\big) \\ \vartheta_2^\varepsilon = 2\big(\partial_1[(K * \mathscr{W}_3{}^\varepsilon - K * \mathscr{W}_3{}^\varepsilon(x))\xi_{21}^\varepsilon] - \partial_2[(K * \mathscr{W}_3{}^\varepsilon - K * \mathscr{W}_3{}^\varepsilon(x))\xi_{11}^\varepsilon]\big) \\ \vartheta_3^\varepsilon = 2\big(\partial_1[(K * \mathscr{W}_1{}^\varepsilon - K * \mathscr{W}_1{}^\varepsilon(x))\xi_{22}^\varepsilon] - \partial_2[(K * \mathscr{W}_1{}^\varepsilon - K * \mathscr{W}_1{}^\varepsilon(x))\xi_{12}^\varepsilon]\big) \end{cases}$$

Lemma

Moreover, there exists a limit $\vartheta := \lim_{\varepsilon\to 0}\vartheta^{\varepsilon}$ *in* $\mathcal{D}'(\mathbb{T}^2)$ *and the components* ϑ_k *of* ϑ, $k = 1,2,3$, *are given by*

$$\begin{cases}\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2)\xi_{23}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\big),\\ \vartheta_2 = 2\big(\partial_1[(K * \mathscr{W}_3)\xi_{21}] - \partial_2[(K * \mathscr{W}_3)\xi_{11}]\big),\\ \vartheta_3 = 2\big(\partial_1[(K * \mathscr{W}_1)\xi_{22}] - \partial_2[(K * \mathscr{W}_1)\xi_{12}]\big),\end{cases}$$

$$\begin{cases}\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{23}] - \partial_2[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{13}]\big),\\ \vartheta_2 = 2\big(\partial_1[(K * \mathscr{W}_3 - K * \mathscr{W}_3(x))\xi_{21}] - \partial_2[(K * \mathscr{W}_3 - K * \mathscr{W}_3(x))\xi_{11}]\big),\\ \vartheta_3 = 2\big(\partial_1[(K * \mathscr{W}_1 - K * \mathscr{W}_1(x))\xi_{22}] - \partial_2[(K * \mathscr{W}_1 - K * \mathscr{W}_1(x))\xi_{12}]\big),\end{cases}$$

where identities are valid for any $x \in \mathbb{T}^2$, *and the products in the square brackets are understood as Young products.*

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

Consider the first component of $\vartheta := \xi \times \overline{\xi}$:

$$\vartheta_1 = 2\big(\partial_1[\underbrace{(K * \mathscr{W}_2)\xi_{23}}_{0^-}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\big)$$

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

Consider the first component of $\vartheta := \xi \times \overline{\xi}$:

$$\vartheta_1 = 2\Big(\partial_1[\underbrace{(K * \mathscr{W}_2)\xi_{23}}_{0^-}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\Big)$$

Remark

We observe that ϑ_1 is well defined since the expression inside the square brackets can be understood in the Young product sense. Moreover, this expression suggests that the expected regularity of ϑ is -1^-. Had this been the case, the expected regularity of $\zeta := 2iG * \vartheta$ would be 0^- and therefore the expected regularity of R would also be 0^-. Such a regularity is too low for the product $R \times \overline{\xi}$ to be well-defined. Hence, a fixed-point argument cannot be realized.

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

Consider the first component of $\vartheta := \xi \times \overline{\xi}$:

$$\vartheta_1 = 2\big(\partial_1[\underbrace{(K * \mathscr{W}_2)\xi_{23}}_{0^-}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\big)$$

Remark

We observe that ϑ_1 is well defined since the expression inside the square brackets can be understood in the Young product sense. Moreover, this expression suggests that the expected regularity of ϑ is -1^-. Had this been the case, the expected regularity of $\zeta := 2iG * \vartheta$ would be 0^- and therefore the expected regularity of R would also be 0^-. Such a regularity is too low for the product $R \times \overline{\xi}$ to be well-defined. Hence, a fixed-point argument cannot be realized.

$$\Gamma_{\varepsilon,\sigma} R := -2G * \Big(R \times \overline{R} + (\gamma\xi) \times \overline{R} + R \times (\gamma\overline{\xi})\Big) - \tilde{\gamma}\zeta + C$$

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

Lemma

Assume that $\eta \in \mathscr{C}^{-\kappa}(\mathbb{T}^2)$, $\kappa \in (0, \frac{1}{2})$, *and let* $\eta^\varepsilon = \eta * \rho_\varepsilon$. *Then, for all* $\varepsilon \geqslant 0$ *and* $j = 1, 2, 3$,

$$\sup_{\substack{x \in \mathbb{T}^2, \lambda \in (0,1], \\ \varphi \in \mathcal{D}(B_1(0)), \|\varphi\|_\infty \leqslant 1}} \frac{\left|\left[(K * \mathscr{W}_j^\varepsilon - K * \mathscr{W}_j^\varepsilon(x))\eta^\varepsilon\right](\varphi_x^\lambda)\right|}{\lambda^{1-\kappa}} \lesssim |K * \mathscr{W}_j^\varepsilon|_{1-\frac{\kappa}{2}} |\eta^\varepsilon|_{-\frac{\kappa}{2}}$$

$$\lesssim |\mathscr{W}_j|_{-1-\frac{\kappa}{2}} |\eta|_{-\frac{\kappa}{2}},$$

where $\mathscr{W}_j^0 := \mathscr{W}_j$, $\eta^0 := \eta$, *and for* $\varepsilon = 0$ *the product on the right-hand side is understood as the Young product.*

Regularity of $\vartheta = \xi \times \overline{\xi}$

Proposition

Let ϑ *be the limit of* $\vartheta^\varepsilon := \xi^\varepsilon \times \overline{\xi^\varepsilon}$ *in* $\mathcal{D}'(\mathbb{T}^2)$. *Then, for every* $\kappa \in (0, \frac{1}{2})$,

$$\vartheta \in E_{-\kappa}.$$

Furthermore, for all $\varepsilon \geqslant 0$, $\kappa \in (0, \frac{1}{2})$, *and for* $j = 1, 2, 3$,

$$\|\vartheta^\varepsilon_j\|_{-\kappa} \lesssim |\mathscr{W}_{j_1}|_{-1-\frac{\kappa}{2}} |\mathscr{W}_{j_2}|_{-1-\frac{\kappa}{2}},$$
$$\|\vartheta^\varepsilon_j - \vartheta_j\|_{-\kappa} \lesssim |\mathscr{W}^\varepsilon_{j_1} - \mathscr{W}_{j_1}|_{-1-\frac{\kappa}{2}} |\mathscr{W}_{j_2}|_{-1-\frac{\kappa}{2}} + |\mathscr{W}^\varepsilon_{j_2} - \mathscr{W}_{j_2}|_{-1-\frac{\kappa}{2}} |\mathscr{W}_{j_1}|_{-1-\frac{\kappa}{2}},$$

where $j_1 := j + 1 \mod 3$ *and* $j_2 := j + 2 \mod 3$, $\vartheta^0 := \vartheta$.

Regularity of $\zeta = 2G * [\xi \times \overline{\xi}]$

COROLLARY

Let ζ *be the limit of* $\zeta^\varepsilon := 2G * \vartheta^\varepsilon$ *in* $\mathcal{D}'(\mathbb{T}^2)$. *Then,*

$$\zeta \in E_{1-\kappa}.$$

Furthermore, for all $\varepsilon \geqslant 0$ *and* $\kappa \in (0, \frac{1}{2})$,

$$\|\zeta^\varepsilon\|_{1-\kappa} \lesssim \|\mathscr{W}\|^2_{-1-\kappa},$$

$$\|\zeta^\varepsilon_j - \zeta_j\|_{1-\kappa} \lesssim |\mathscr{W}^\varepsilon_{j_1} - \mathscr{W}_{j_1}|_{-1-\frac{\kappa}{2}} |\mathscr{W}_{j_2}|_{-1-\frac{\kappa}{2}} + |\mathscr{W}^\varepsilon_{j_2} - \mathscr{W}_{j_2}|_{-1-\frac{\kappa}{2}} |\mathscr{W}_{j_1}|_{-1-\frac{\kappa}{2}},$$

where $j = 1, 2, 3$; $j_1 := j + 1 \mod 3$; $j_2 := j + 2 \mod 3$; $\zeta^0 := \zeta$.

Regularity of $\vartheta = \xi \times \overline{\xi}$

Recall:

$$\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2)\xi_{23}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\big),$$

$$\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{23}] - \partial_2[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{13}]\big).$$

The latter representation holds for any $x \in \mathbb{T}^2$.

Regularity of $\vartheta = \xi \times \overline{\xi}$

Recall:

$$\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2)\xi_{23}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\big),$$

$$\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{23}] - \partial_2[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{13}]\big).$$

The latter representation holds for any $x \in \mathbb{T}^2$.

For the proof we will need

$$\tilde{\mathscr{B}}^1 := \{\varphi \in \mathcal{D}(B_1(0)) : \|\partial^k \varphi\|_\infty \leqslant 1,\ |k| = 0, 1\},$$

$$\tilde{\mathscr{B}}^0 := \{\varphi \in \mathcal{D}(B_1(0)) : \|\varphi\|_\infty \leqslant 1\}.$$

Regularity of $\vartheta = \xi \times \overline{\xi}$

Recall:

$$\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2)\xi_{23}] - \partial_2[(K * \mathscr{W}_2)\xi_{13}]\big),$$

$$\vartheta_1 = 2\big(\partial_1[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{23}] - \partial_2[(K * \mathscr{W}_2 - K * \mathscr{W}_2(x))\xi_{13}]\big).$$

The latter representation holds for any $x \in \mathbb{T}^2$.

For the proof we will need

$$\tilde{\mathscr{B}}^1 := \{\varphi \in \mathcal{D}(B_1(0)) : \|\partial^k \varphi\|_\infty \leqslant 1,\ |k| = 0, 1\},$$

$$\tilde{\mathscr{B}}^0 := \{\varphi \in \mathcal{D}(B_1(0)) : \|\varphi\|_\infty \leqslant 1\}.$$

Recall: $\xi_3 := G * \mathscr{W}_3$, $\xi_{23} := \mathrm{Re}\,\xi_3$, $\xi_{13} := \mathrm{Im}\,\xi_3$.

Regularity of $\vartheta = \xi \times \overline{\xi}$

Outline of proof

$$|\vartheta_1^\varepsilon|_{-\kappa} = \sup_{\substack{x\in\mathbb{T}^2,\,\lambda\in[0,1]_{\mathbb{D}},\\ \varphi\in\mathscr{B}^1}} \frac{|\vartheta_1^\varepsilon(\varphi_x^\lambda)|}{\lambda^{-\kappa}} \simeq \sup_{\substack{x\in\mathbb{T}^2,\,\lambda\in(0,1],\\ \varphi\in\tilde{\mathscr{B}}^1}} \frac{|\vartheta_1^\varepsilon(\varphi_x^\lambda)|}{\lambda^{-\kappa}}$$

$$= \sup_{\substack{x\in\mathbb{T}^2,\,\lambda\in(0,1],\\ \varphi\in\tilde{\mathscr{B}}^1}} \lambda^\kappa \Big| \partial_2 \Big[\big(K * \mathscr{W}_1{}^\varepsilon - K * \mathscr{W}_1{}^\varepsilon(x)\big)\xi_{23}^\varepsilon\Big](\varphi_x^\lambda)$$

$$- \partial_1 \Big[\big(K * \mathscr{W}_1{}^\varepsilon - K * \mathscr{W}_1{}^\varepsilon(x)\big)\xi_{13}^\varepsilon\Big](\varphi_x^\lambda)\Big|$$

$$\leqslant \sup_{\substack{x\in\mathbb{T}^2,\,\lambda\in(0,1],\\ \varphi\in\tilde{\mathscr{B}}^1}} \frac{\Big|\Big[\big(K * \mathscr{W}_1{}^\varepsilon - K * \mathscr{W}_1{}^\varepsilon(x)\big)\xi_{23}^\varepsilon\Big](\partial_1\varphi)_x^\lambda\Big|}{\lambda^{1-\kappa}}$$

$$+ \sup_{\substack{x\in\mathbb{T}^2,\,\lambda\in(0,1],\\ \varphi\in\tilde{\mathscr{B}}^1}} \frac{\Big|\Big[\big(K * \mathscr{W}_1{}^\varepsilon - K * \mathscr{W}_1{}^\varepsilon(x)\big)\xi_{13}^\varepsilon\Big](\partial_2\varphi)_x^\lambda\Big|}{\lambda^{1-\kappa}}.$$

Outline of Proof

Evaluating the first term:

$$\sup_{\substack{x\in\mathbb{T}^2,\lambda\in(0,1],\\ \varphi\in\tilde{\mathscr{B}}^1}} \frac{\left|\left[\left(K * \mathscr{W}_1{}^{\varepsilon} - K * \mathscr{W}_1{}^{\varepsilon}(x)\right)\xi_{23}^{\varepsilon}\right](\partial_1\varphi)_x^{\lambda}\right|}{\lambda^{1-\kappa}}$$

$$\leqslant \sup_{\substack{x\in\mathbb{T}^2,\lambda\in(0,1],\\ \psi\in\tilde{\mathscr{B}}^0}} \frac{\left|\left[\left(K^{\varepsilon} * \mathscr{W}_1 - K^{\varepsilon} * \mathscr{W}_1(x)\right)\xi_{23}^{\varepsilon}\right]\psi_x^{\lambda}\right|}{\lambda^{1-\kappa}}$$

$$\leqslant |K * \mathscr{W}_1{}^{\varepsilon}|_{1-\frac{\kappa}{2}}|\xi_{23}^{\varepsilon}|_{-\frac{\kappa}{2}} \leqslant |K * \mathscr{W}_1|_{1-\frac{\kappa}{2}}|\xi_{23}|_{-\frac{\kappa}{2}} \lesssim |\mathscr{W}_1|_{-1-\frac{\kappa}{2}}|\mathscr{W}_2|_{-1-\frac{\kappa}{2}}.$$

Rate of convergence

Lemma

The following inequalities are satisfied for all $\varepsilon > 0$ *and* $\kappa \in (0, \frac{1}{2})$:

$$\|\xi^{\varepsilon} - \xi\|_{-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|^2_{-1-\kappa},$$

$$\|\vartheta^{\varepsilon} - \vartheta\|_{-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|^2_{-1-\kappa},$$

$$\|\zeta^{\varepsilon} - \zeta\|_{1-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|^2_{-1-\kappa}.$$

Rate of convergence

Lemma

The following inequalities are satisfied for all $\varepsilon > 0$ *and* $\kappa \in (0, \frac{1}{2})$:

$$\|\xi^\varepsilon - \xi\|_{-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|^2_{-1-\kappa},$$

$$\|\vartheta^\varepsilon - \vartheta\|_{-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|^2_{-1-\kappa},$$

$$\|\zeta^\varepsilon - \zeta\|_{1-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|^2_{-1-\kappa}.$$

The proof follows from

$$|\mathscr{W}_j^\varepsilon - \mathscr{W}_j|_{-1-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} |\mathscr{W}_j|_{-1-\frac{\kappa}{2}}, \quad j = 1, 2, 3, \quad \text{a.s.}$$

Rate of convergence

Lemma

Let $\delta > 0$, $\kappa \in (0, \frac{1}{2})$, $\Omega_\delta \subset \Omega$, *and* $\sigma, \varsigma > 0$ *be the same as previously constructed, so that the maps* $\Gamma_{\varepsilon,\sigma}$ *and* $\Gamma_{0,\sigma}$ *possess unique fixed points* R^ε *and, respectively,* R *in* $M_{\sigma,1-\kappa}$ *for* $|\gamma| < \varsigma$. *Then, there exists a constant* $C_\delta > 0$, *depending only on* δ, *such that for all* $\omega \in \Omega_\delta$,

$$\|R - R^\varepsilon\|_{1-\kappa} \leqslant C_\delta \, \varepsilon^{\frac{\kappa}{2}},$$

$$\|r - r^\varepsilon\|_{-\kappa} \leqslant C_\delta \, \varepsilon^{\frac{\kappa}{2}},$$

where $r := R - \gamma\xi$ *and* $r^\varepsilon := R^\varepsilon - \gamma\xi^\varepsilon$.

Main result

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3)$ *be a 3D white noise on* $\mathbb{T}^2$ *such that* $\mathscr{W}_3$ *has zero mean over* $\mathbb{T}^2$*. Assume that* $\kappa \in (0, \frac{1}{2})$*. Then, for every* $\delta > 0$*, there exists a number* $\varsigma > 0$ *and a set* $\Omega_\delta \subset \Omega$*, depending also on* κ*, with* $\mathbb{P}(\Omega_\delta) > 1 - \delta$*, such that for every* $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ *with* $|\gamma| < \varsigma$*, and for every* $\varepsilon > 0$*, the system*

$$\partial_{\bar{z}} r^\varepsilon = r^\varepsilon \times \overline{r^\varepsilon} + i\,\gamma\,\mathscr{W}^\varepsilon,$$

considered for $\omega \in \Omega_\delta$*, has a solution* $r^\varepsilon \in \mathrm{C}^\infty(\mathbb{T}^2, \mathbb{C}^3)$ *such that*

$$\|r^\varepsilon - r\|_{-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}} \quad \textit{uniformly in } \ \omega \in \Omega_\delta.$$

This talk is based on the paper

Z. Brzeźniak, M. Neklyudov, E. Shamarova, Singular SPDEs with the Cauchy-Riemann operator on a torus, arXiv:2503.20075, 2025

Thank you!