Bismut-Elworthy type formulae for BSDEs with degenerate noise

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joint works with
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Outline

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Forward SDE in H

SDE in the Hilbert space H:

$$\begin{cases} dX_t = AX_t dt + B(t, X_t)dt + G(t, X_t)dW_t, & t \in [s, T] \subset [0, T], \\ X_\tau = x, & \tau \in [0, s] \end{cases}$$

- A: generator of a C_0 semigroup in H
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- B, G: Lipschitz continuous and differentiable w.r. to x
- transition semigroup $P_{s,t} : B_b(H) \to B_b(H)$ associated to X:

$$P_{s,t} f(x) = \mathbb{E}[f(X_t^{s,x})], \forall f \in B_b(H), x \in H$$

- G is invertible and $|G^{-1}(t, x)|_{L(H,H)} \le C$

Bismut-Elworthy formula for SDEs

- $P_{s,t}f \in C_b^1(H)$ if $f \in C_b^1(H)$:

$$\langle (\nabla P_{s,t}f)(x), h\rangle_H = \mathbb{E}[\nabla f(X_t^{s,x})\nabla_x X_t^x h]$$
$$= \mathbb{E}[\nabla f(X_t^{s,x})\langle DX_t^{s,x}, \widetilde{u}_t\rangle_{L^2(0,T;U)}] = \mathbb{E}[\langle D(f(X_t^{s,x})), \widetilde{u}_t\rangle_{L^2(0,T;H)}]$$

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- by an approximation procedure $\forall\, f \in B_b(H)$

$$\langle (\nabla_x P_{s,t}f)(x), h\rangle_H$$
$$= \mathbb{E}[f(X_t^x)\frac{1}{t-s}\int_s^t \left\langle G(r, X_r^{s,x})^{-1}\nabla_x X_r^{s,x}\, h,\, dW_s\right\rangle .], \qquad t \in (0, T].$$

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- $|\langle (\nabla_x P_{s,t} f)(x), h\rangle_H|$

$$\leq \|f\|_\infty \mathbb{E}[| \frac{1}{t-s} \int_s^t \left\langle G(r, X_r^{s,x})^{-1} \nabla_x X_r^{s,x}\, h,\, dW_s \right\rangle |] \leq \frac{C}{t-s} \|f\|_\infty |h|_H$$

see e.g. Da Prato-Zabczyk Book 3, ...Cerrai ...

BSDE

Backward equation coupled with forward process:

$$\begin{cases} dX_t = AX_t dt + B(t, X_t)dt + G(t, X_t)dW_t, & t \in [s, T] \subset [0, T], \\ X_s = x, & \tau \in [0, s], \\ -dY_t = \psi(t, X_t, Y_t, Z_t)\, dt - Z_t\, dW_t, & t \in [0, T], \\ Y_T = \phi(X_T), \end{cases}$$

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- the solution of the BSDE is the pair of processes (Y, Z)
  the solution exist under Lipschitz assumptions on ψ in (Y, Z)
- if $\psi \equiv 0$, $Y_t^{s,x} = \mathbb{E}[\phi(X_T^{s,x})]$

Connections with Kolmogorv equations

Linear Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t,x) = \mathcal{L}_t v\,(t,x)\,, & t \in [0,T]\,,\ x \in H \\ v(T,x) = \phi\,(x)\,, \end{cases}$$

- $\mathcal{L}_t$: generator of $P_{t,\cdot}$

$$(\mathcal{L}_t f)(x) = \frac{1}{2}(TrG(t,x)G^*(t,x)\nabla^2 f)(x) + \langle Ax, \nabla f(x)\rangle + \langle B(t,x), \nabla f(x)\rangle.$$

- $u(t,x) = \mathbb{E}[\phi(X_T^{t,x}) = P_{t,T}[\phi](x)]$

Connections with (semilinear) Kolmogorv equations

Semilinear Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t,x) = \mathcal{L}_t v\,(t,x) + \psi\,(t,x,v(t,x),\nabla v(t,x)G(t,x))\,, \quad t \in [0,T] \\ v(T,x) = \phi\,(x)\,, \end{cases}$$

- $\mathcal{L}_t$ as before
- $u(t,x) = \mathbb{E}[\phi(X_T^{t,x}) + \int_t^T \psi(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\, dr]$ defined via Y from the BSDE
- $u(t,x) := Y_t^{t,x}$

Bismut–Elworthy Formula (Nonlinear Case)

$$U_t^{h,s,x} = \frac{1}{t-s} \int_t^\tau \left\langle G(s, X_r^{s,x})^{-1} \nabla_x X_r^{s,x} \, h, \, dW_r \right\rangle .$$

Under invertibility of $G(t, x)$:

$$\nabla_x Y_t^{s,x} h = \mathbb{E}\left[\int_t^T \psi(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}) U_r^{h,s,x} dr + \phi(X_T) U_T^{h,s,x} \right]$$

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- Formula yields a-priori estimates on $\nabla_x Y_t^{s,x}$
- Enables definition of solution $u(t,x) := Y_t^{t,x}$ for nonlinear Kolmogorov equations
- Regularity of u despite possibly non-differentiable data

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- The diffusion G is such that $e^{tA}G(\tau) \in \mathcal{L}_2(U; H)$

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- Derivative of $X^{s,x}$: $\nabla_x X_t^{s,x}h = h$, $0 \leq t \leq s$ and satisfies

$$\nabla_x X_t^{s,x}h = e^{(t-s)A}h + \int^t e^{(t-\sigma)A}\nabla_x B(\sigma, X_\sigma^{s,x})\nabla_x X_\sigma^{s,x} h d\sigma,$$

Malliavin Derivative of X_t

$X^{s,x} \in \mathbb{L}^{1,2}(H)$ and there exists a version of $DX^{s,x}$ s.t. $(D_\tau X_t^{s,x})_{t\in(\tau,T]}$ satisfies for $0 \le \tau < t \le T$

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$$D_\tau X_t^{s,x} = 0, \qquad \text{for } 0 \leq t \leq \tau < T.$$

and $\forall\, \widetilde{u} \in L^2(\Omega \times [0,T]; U)$

$$\langle DX_t^{s,x}, \widetilde{u}\rangle_{L^2(0,T;U)} = \int_0^T D_\tau X_t^{s,x} \widetilde{u}(\tau) d\tau$$

$$= \int_s^t e^{(t-r)A}\nabla_x B(r, X_r^{s,x}) \langle DX_r^{s,x}, \widetilde{u}\rangle_{L^2(0,T;U)} dr + \int_s^t e^{(t-\tau)A} G(\tau)\widetilde{u}(\tau)\tau.$$

Classical vs Malliavin derivative

Assume $B = 0$: LINEAR CASE

$$\nabla_x X_t^{s,x} h = e^{(t-s)A} h$$

$$\langle D X_t^{s,x}, \widetilde{u} \rangle_{L^2(0,T;U)} = \int_s^t e^{(t-\tau)A} G(\tau) \widetilde{u}(\tau) dr.$$

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$$\langle DX_t^{s,x}, \widetilde{u} \rangle_{L^2(0,T;U)} = \int_s^t e^{(t-\tau)A} G(\tau) \widetilde{u}(\tau) dr.$$

If we find a family $(\widetilde{u}_t)_{t \in (s,T]} \subseteq L^2([0,T];U)$ such that for every $t \in (s,T]$ we get

$$\int_s^t e^{(t-\tau)A} G(\tau) \widetilde{u}_t(\tau) d\tau = e^{(t-s)A} h, \quad \mathbb{P}\text{-a.s},$$

then

$$\nabla_x X_t^{s,x} h = \langle DX_t^{s,x}, \widetilde{u}_t \rangle_{L^2(0,T;U)}$$

$(\widetilde{u}_t)_{t \in (s,T]}$ also depends on s, x and h: $\widetilde{u}_t = \widetilde{u}_{h,s,x,t}$.

Bismut–Elworthy Formula (Linear Case)

- $P_{s,t}[f] \in C_b^1(H)$ if $f \in C_b^1(H)$:

$$\begin{aligned}\langle (\nabla_x P_{s,t}[f])(x), h \rangle_H &= \mathbb{E}[\nabla f(X_t^x) \nabla_x X_t^{s,x} h] \\ &= \mathbb{E}[\nabla f(X_t^{s,x}) \langle DX_t^{s,x}, \widetilde{u}_t \rangle_{L^2(0,T;U)}] \\ &= \mathbb{E}[\langle D(f(X_t^{s,x})), \widetilde{u}_t \rangle_{L^2(0,T;U)}]\end{aligned}$$

chain rule: $D(f(X_t^{s,x})) = \nabla f(X_t^x) DX_t^{s,x}$.

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- If $\widetilde{u}_t \in \mathrm{dom}(\delta)$ integrating by parts

$$\langle (\nabla_x P_{s,t}[f])(x), h\rangle_H = \mathbb{E}[f(X_t^x)\delta(\widetilde{u}_t)], \qquad t \in (0, T].$$

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- by an approximation procedure still valid $\forall f \in B_b(H)$

$$|\langle (\nabla_x P_{s,t}[f])(x), h\rangle_H| \le \|f\|_\infty \|\widetilde{u}_t\|_{L^2([0,T];U)}.$$

FBSDE

$$
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with $B(t, X_t) = G(t)\bar{B}(t, X_t)$; if

$$|\psi(t, x, y_1, z_1) - \psi(t, x, y_2, z_2)| \leq L_\psi \left(+|y_1 - y_2| + |z_1 - z_2|\right),$$
$$|\psi(t, x, 0, 0)| \leq K_\psi(1 + |x|^m), \qquad |\phi(x)| \leq K_\phi(1 + |x|^m),$$

then $\exists\,!$ solution Y, Z s.t.

$$\|Y\|_{\mathcal{S}^p} + \|Z\|_{\mathcal{M}^p} := \mathbb{E}[\sup_{s \in [0,T]} |Y_s|^p] + \mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}} \leq C(1 + |x|^m)$$

Bismut Formula for FBSDE

Assume that $G(t) = G\widetilde{G}(t)$, $G \in \mathcal{L}(U, H)$, $\widetilde{G} : U \to U$

$$\|\widetilde{u}_{h,s,x,r}\|_{L^2(s,r;H)} \leq \frac{C\|h\|_H}{(r-s)^\alpha}, \qquad 0 \leq s < r < T,\ \alpha \in (0,1)$$

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Assume $\int_s^t e^{(t-\tau)A} G(\tau)\widetilde{u}_{h,s,x,t}(\tau)d\tau = e^{(t-s)A}h,$

Bismut formula

$$\begin{aligned}\mathbb{E}\left[\langle \nabla_x Y_t^{s,x}, h\rangle\right] &= \mathbb{E}\int_t^T \psi\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) \int_s^r \langle \widetilde{u}_{h,s,x,r}(\sigma), dW_\sigma\rangle_U \, dr \\ &+ \mathbb{E}\int_t^T Z_r \bar{B}(r, X_r^{s,x}) \int_s^r \langle \widetilde{u}_{h,s,x,r}(\sigma), dW_\sigma\rangle_U dr \\ &+ \mathbb{E}\left[\phi(X_T^{s,x}) \int_s^T \langle \widetilde{u}_{h,s,x,T}(\sigma), dW_\sigma\rangle_U\right].\end{aligned}$$

Idea of the proof

- ϕ, ψ and $\bar{B}$ differentiable with respect to x, y and z.
- $\bar{B} = 0$. BSDE in integral form: for $t \in [s, T]$

$$Y_t^{s,x} = \int_t^T \psi(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x})\, dr + \int_t^T Z_r\, dW_r + \phi(X_T^{s,x})$$

- $\forall t \in [s, T]$, $\xi \in H$, $(t, \xi) \mapsto Y_t^{t,\xi}$ is deterministic: set

$$v(t,\xi) := Y_t^{t,\xi}, \quad Z_t^{t,\xi} := \bar{v}(t,\xi)$$

- Use the Markov property and

$$\langle \nabla_x X_r^{s,x}, h \rangle = \int_s^r D_\sigma X_r^{s,x} \widetilde{u}_{h,s,x,r}(\sigma) d\sigma, \qquad r \in (s, T]$$

$$D_\sigma Y_r^{s,x} = D_\sigma v(r, X_r^{s,x}) = \nabla_\xi v(r, X_r^{s,x}) D_\sigma X_r^{s,x}$$

$$D_\sigma Z_r^{s,x} = D_\sigma \bar{v}(r, X_r^{s,x}) = \nabla_\xi \bar{v}(r, X_r^{s,x}) D_\sigma X_r^{s,x}$$

Idea of the proof II

- we get

$$\mathbb{E}\langle \nabla_x Y_t^{s,x}, h\rangle = \mathbb{E}\phi(X_T^{s,x}) \int_s^T \langle \widetilde{u}_{h,s,x,T}(\sigma),\ dW_\sigma\rangle_U$$
$$+ \mathbb{E}\int_t^T \psi(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}) \int_s^r \langle \widetilde{u}_{h,s,x,r}(\sigma), dW_\sigma\rangle_U dr$$

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-

$$|\langle \nabla_x Y_s^{s,x}, h\rangle| \leq \frac{C(1+|x|^m)|h|_H}{(T-s)^\alpha}, \qquad s \in [0,T),\ a \in U.$$

Idea of the proof III

- if $\bar{B} \neq 0$ we set $\widetilde{\mathbb{P}} := \Psi \mathbb{P}$ and $\widetilde{W_t} = W_t + \int_0^t \bar{B}(s, X^{s,x})\, ds$.

$$\Psi := \exp\left\{ \int_s^T \left\langle \bar{B}\left(r, X_r^{s,x}\right), dW_r \right\rangle_U - \frac{1}{2} \int_s^T \left\| \bar{B}\left(r, X_r^{s,x}\right) \right\|_U^2 ds \right\},$$

Idea of the proof III

- if $\bar{B} \neq 0$ we set $\widetilde{\mathbb{P}} := \Psi \mathbb{P}$ and $\widetilde{W_t} = W_t + \int_0^t \bar{B}(s, X^{s,x})\, ds$.

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- under $\widetilde{P}$ the FBSDE can be written as

$$\begin{cases} dX_t = AX_t dt + G(t) d\widetilde{W}_t, \\ X_\tau = x, \\ -dY_t = \psi(t, X_t, Y_t, Z_t)\, dt + Z_t \bar{B}(t, X_t)\, dt - Z_t\, d\widetilde{W}_t, \\ Y_T = \phi(X_T). \end{cases}$$

Outline

Bismut formula for BSDEs: invertible diffusion

Bismut formula for BSDEs: possibly degenerate diffusion

Applications

Perspectives and Bibliography

Abstract Stochastic Wave Equation

crucial point: find $(\widetilde{u}_{h,s,x,t})_{t\in(s,T]} \subseteq L^2([0,T];U)$ s.t. $\forall\, t \in (s,T]$

$$\int_s^t e^{(t-\tau)A} G(\tau)\widetilde{u}_t(\tau)d\tau = e^{(t-s)A}h, \quad \mathbb{P}\text{-a.s},$$

Stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial\tau^2} y(\tau,\xi) = \frac{\partial^2}{\partial\xi^2} y(\tau,\xi) + \dot{W}(\tau,\xi), \\ y(\tau,0) = y(\tau,1) = 0, \\ y(0,\xi) = x_0(\xi), \ \frac{\partial y}{\partial\tau}(0,\xi) = x_1(\xi), \quad \tau \in (0,T], \quad \xi \in [0,1], \end{cases}$$

Abstract Stochastic Wave Equation

$$\begin{cases} \frac{d^2y}{dt^2}(t) = \Lambda y(t) + \sigma \dot{W}(t), \\ y(0) = x_0, \\ \frac{dy}{dt}(0) = x_1, \quad t \in (0,T], \end{cases}$$

$A = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}$ in $\mathcal{H} = H_0^1([0,1]) \times L^2([0,1]) = \mathcal{D}(\Lambda^{1/2}) \times U$

Λ positive self-adjoint operator on U

$$dX_t^{0,x} = AX_t^{0,x}dt + GdW_t, \ t \in [0,T], \ X_0^{0,x} = x \in H := U \times V',$$

H vs $\mathcal{H} = V \times U$: the operators $Q_\tau = \int_0^\tau e^{sA}GG^* e^{sA^*}ds, \quad \tau \geq 0,$

are of trace class from H into H but not from $\mathcal{H}$ into $\mathcal{H}$

$$Gu := \begin{pmatrix} 0 \\ \tilde{G}u \end{pmatrix}, \qquad \tilde{G} \text{ invertible}$$

Directional differentiability

- $F : H \to \mathbb{R}$ s differentiable along $\mathcal{H} = V \times U \subseteq H$ if

$$\lim_{s \to 0} \frac{F(x + sk) - F(x)}{s} := \nabla_k F(x) = \langle \nabla^{\mathcal{H}} F(x), k \rangle_{\mathcal{H}}, \; k \in \mathcal{H}$$

- Differentiability along the direction $Jb = \begin{pmatrix} 0 \\ b \end{pmatrix} \in \mathcal{H} \subseteq H$, $b \in U$, $J : U \to H$

$$\nabla_b^J F(x) = \nabla_{Jb} F(x), \quad b \in U, \; x \in H,$$

Wave transition semigroup

$P_t\,[f]$, $t > 0$, $f \in B_b(H)$ wave transition semigroup applied to f.
$\forall\, k \in \mathcal{H}\ \exists\ \widetilde{u} \in L^2_{\mathcal{P}}(\Omega \times [0, T]; U)$ such that

$$\nabla_k(P_t[f])(x) = \langle \nabla^{\mathcal{H}}(P_t[f])(x), k \rangle_{\mathcal{H}} = \mathbb{E}[f(X_t^x) \int_0^t \widetilde{u}_s dW_s]$$

and for $k \in \mathcal{H}$

$$|\langle \nabla^{\mathcal{H}}(P_t[f])(x), k \rangle_{\mathcal{H}}| \leq \frac{c}{t^{3/2}} \|f\|_\infty$$

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and for $k \in \mathcal{H}$

$$|\langle \nabla^{\mathcal{H}}(P_t[f])(x), k\rangle_{\mathcal{H}}| \leq \frac{c}{t^{3/2}}\|f\|_\infty$$

Moreover if $h = Jb = \begin{pmatrix} 0 \\ b \end{pmatrix}$ then

$$|\langle \nabla^{\mathcal{H}}(P_t[f])(x), \rangle_{\mathcal{H}}| = |\nabla^J_b(P_t[f])(x)| \leq \frac{c}{t^{1/2}}\|f\|_\infty$$

Proof of differentiability for the wave transition semigroup

$$\int_s^t e^{(t-r)A} G \widetilde{u}(r) dr = e^{(t-s)A} h, \quad \mathbb{P}\text{-a.s.},$$

consider $h = \begin{pmatrix} a \\ b \end{pmatrix} \in V \times U$ and set

$$\widetilde{u}(s) = \widetilde{G}^{-1} \psi_1(s) + \widetilde{G}^{-1} \psi_2'(s) \ \in U$$

where

$$\psi_1(s) = \Phi_t(s)[-\Lambda^{1/2} \sin(\Lambda^{1/2} s) a + \cos(\Lambda^{1/2} s) b],$$

$$\psi_2(s) = \Phi_t(s)[\cos(\Lambda^{1/2} s) a + \frac{1}{\Lambda^{1/2}} \sin(\Lambda^{1/2} s) b],$$

and

$$\Phi_t(s) = \frac{s^2 (t-s)^2}{\int_0^t r^2 (t-r)^2 dr} \qquad s \in [0, t].$$

Going on with the proof

$\widetilde{u}$ well defined in U $\widetilde{G}^{-1} \in L(U, U)$ and both $\psi_1(s), \psi_2'(s) \in U$.

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$$
\begin{aligned}
&\int_0^t e^{(t-s)A} G\widetilde{u}(s)ds \\
=&\int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ \psi_1(s) \end{pmatrix} ds + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ \psi_2'(s) \end{pmatrix} ds \\
=&\int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ \psi_1(s) \end{pmatrix} ds + \int_0^t e^{(t-s)A} A \begin{pmatrix} 0 \\ \psi_2(s) \end{pmatrix} ds \\
=&\int_0^t e^{(t-s)A} \begin{pmatrix} \psi_2(s) \\ \psi_1(s) \end{pmatrix} ds = \int_0^t e^{(t-s)A} e^{sA} h\Phi(s)ds \\
=&e^{tA}h
\end{aligned}
$$

Going on with the proof

L^2-norm of $\widetilde{u}$ in $\Omega \times (0,t)$. We have

$$\begin{aligned}\|\widetilde{u}\|^2_{L^2((0,t);U)} \leq & 2\|\widetilde{G}^{-1}\|_\infty \left(\int_0^t |\psi_1(s)|^2_U ds + \int_0^t |\psi_2'(s)|^2_U ds\right) \\ \leq & 2\widetilde{C}\|\widetilde{G}^{-1}\|_\infty \left(t^{-1}(|\Lambda^{1/2}a|^2_U + |b|^2_U) + t^{-3}(|\Lambda^{1/2}a|^2_U + |b|^2_U)\right. \\ \leq & \widetilde{M}\frac{|h|^2_{V\times U}}{t^3},\end{aligned}$$

since $a \in V = \Lambda^{-1/2}(U)$, that $|\Lambda^{1/2}a|_U = |a|_V$ and that $|h|^2_{V\times U} = |a|^2_V + |b|^2_U$.

Going on with the proof

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since $a \in V = \Lambda^{-1/2}(U)$, that $|\Lambda^{1/2}a|_U = |a|_V$ and that $|h|^2_{V\times U} = |a|^2_V + |b|^2_U$. If $h = \begin{pmatrix} 0 \\ b \end{pmatrix} \in H$ with $b \in U$, then arguing as above we get

$$\|\widetilde{u}\|^2_{L^2((0,t);U)} \leq \widetilde{M}\frac{|b|_U}{t},$$

foe some $\widetilde{M} > 0$

Outline

Conclusions and future developments

Up to now we have considered

- Bismut–Elworthy formulae for evolution equations with degenerate noise
- Wave equations and damped wave equations
- Non linear case $\rightsquigarrow$ framework suitable for further applications in stochastic control.

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future work

- study the quadratic case: ψ in the BSDE with quadratic growth w.r. to Z
- consider multiplicative noise

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Thank you for your time!