On the Rearranged stochastic heat equation

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Conference 'IndaM' Irregular Stochastic Analysis

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Joint works W. Hammersley (ex: Nice) and R. Likibi Pellat (Nice)

1. Motivation

Background

- Well-known fact in stochastic analysis:

$$\dot{X}_t = b_t(X_t)$$

  - b continuous $\Rightarrow$ existence but ~~uniqueness~~
  - restore uniqueness by adding a Brownian motion $(B_t)_{t \geq 0}$

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- Picture for McKean-Vlasov equations

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with $\mathcal{L}(\cdot)$ standing for the law

  - typically require Lipschitz continuity in μ (e.g. w.r.t TV)

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- Program
  - random forcing for

$$\partial_t \mu_t = -\text{div}(b_t(\cdot, \mu_t)\mu_t)$$

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Semi-group interpretation

- Semi-group generated by Fokker-Planck equation

$$\partial_t \mu_t + \mathrm{div}\Big(b(\cdot, \mu_t)\mu_t\Big) - \Delta \mu_t = 0$$

  - if well-posed, let for a test function $\phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$,

$$\mathscr{P}_t \phi : \mu_0 \mapsto \phi(\mu_t)$$

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  - get a transport PDE for $\mathcal{U}(t, \mu) = \mathscr{P}_t \phi(\mu)$

$$\partial_t \mathcal{U}(t, \mu) = \int_{\mathbb{R}^d} b(y, \mu) \cdot \nabla \mathcal{U}(t, \mu, y) d\mu(y) + \int_{\mathbb{R}^d} \mathrm{div}_y \nabla \mathcal{U}(t, \mu, y) d\mu(y)$$

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- Derivative [Albeverio et al., Ambrosio et al., Dawson, Lions...]
  - for ϕ smooth function of μ

$$\nabla \phi(m,y) = \frac{d}{dy}\Big[\frac{d}{ds}_{|s=0+} \phi\Big(s\delta_y + (1-s)m\Big)\Big]$$

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- What about 'true diffusive equations'? (useful for nonlinear ones)

2. Several Candidates

Form of the noise

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- Works in higher dimension

  ◦ Albeverio, Kondratiev and Röckner [98], when working on the space of configurations (locally finite sum of δ masses)

  ◦ Dello Schiavo [20,25], Sturm [24] on $\mathcal{P}(\mathbb{R}^d)$ (or $\mathcal{P}(\mathbb{T}^d)$)

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- Here, more interested in smoothing $\rightsquigarrow$ two routes

  ◦ infinite-dimensional particle system ([work with Martini and Sodini based on Dello-Schiavo's own works]), $d \geq 2$

  ◦ rearranged SHE ([work with Hammersley]), $d = 1$

Dirichlet-Fergusson process [following Dello Schiavo]

- Infinite but countable particle system
  - flow of purely atomic measures

$$\mu_t := \sum_{i\geq 1} s_i \delta_{X_t^i}, \quad t \geq 0; \qquad dX_t^i = \frac{1}{\sqrt{s_i}} dB_t^i, \quad i \in \mathbb{N}^*$$

where $s_1 > s_2 > \cdots \geq 0$ with $\sum_{i=1}^{+\infty} s_i = 1$ and $\{B^i\}_{i=1}^{\infty}$ collection of d-dimensional independent Brownian motions

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- Main idea: randomize the weights $s_1 > s_2 > \cdots \geq 0$ and the initial conditions $(X_0^1, X_0^2, \cdots)$
  - in other words, looks for a 'convenient' probability measure on atomic probability measures

Energy

• With $\mathfrak{m}$, probability measure on $\mathcal{P}(\mathbb{T}^d)$, associate quadratic form (or energy), for smooth functions φ,

$$\mathcal{E}(\varphi) = \frac{1}{2} \int_{\mathcal{P}(\mathbb{T}^d)} \left\| \boldsymbol{\nabla} \varphi(\mu) \right\|^2_{L^2(\mathbb{T}^d, \mu)} d\mathfrak{m}(\mu)$$

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• Main question: identify $\mathfrak{m}$ such that energy is a Dirichlet form associated with the operator $\boldsymbol{\Delta}$ of the particle system:

$$\int_{\mathcal{P}(\mathbb{T}^d)} \boldsymbol{\Delta}\mathcal{U}(\mu)\mathcal{V}(\mu) d\mathfrak{m}(\mu)$$

$$= - \int_{\mathcal{P}(\mathbb{T}^d)} \left[\int_{\mathbb{T}^d} \nabla \mathcal{U}(\mu, y) \cdot \nabla \mathcal{V}(\mu, y) d\mu(y) \right] d\mathfrak{m}(\mu)$$

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• Makes it possible to define Sobolev H^1 (derivatives in $L^2(\mathfrak{m})$) and then solve in $C([0, T]; L^2) \cap L^2([0, T]; H^1)$

$$\partial_t \mathcal{U}_t + b \cdot \boldsymbol{\nabla}\mathcal{U}_t + \frac{1}{2}\boldsymbol{\Delta}\mathcal{U}_t = 0 \quad \text{for } t \in (0, T), \quad u_t\big|_{t=T} = g$$

3. Rearranged Noise

General plan

- Follow P.L. Lions' approach to differential calculus on $\mathcal{P}_2(\mathbb{R})$
  - see function $\varphi : \mathcal{P}_2(\mathbb{R}) \ni \mu \mapsto \varphi(\mu) \in \mathbb{R}$ as

$$L^2(\mathbb{S} = \mathbb{R}/\mathbb{Z}, dx) \ni X \mapsto \varphi(\mathcal{L}(X))$$

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- Move according to

$$dX_t(x) = \Delta X_t(x)dt + dW_t(x), \quad x \in \mathbb{S}, \ t \geq 0,$$

  - with

$$W_t(x) = \sum_{m \in \mathbb{Z}} W_t^m e_m(x)$$

where $((W_t^m)_{t \geq 0})_{m \in \mathbb{Z}}$ are $\perp\!\!\!\perp$ Brownians and $(e_m)_{m \in \mathbb{Z}}$ is Fourier basis

- Recall the shape of the solution

$$X_t(x) = \left[\exp(t\Delta)X_0 + \int_0^t \exp((t-s)\Delta)dW_s \right](x)$$

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$$X_t \rightsquigarrow \left[\exp(dt\Delta)X_t + \int_0^{dt} \exp((dt-s)\Delta)dW_{t+s} \right] \rightsquigarrow \text{re-arrangement} = X_{t+dt}$$

in order to make it intrinsic $\rightsquigarrow$ RE-ARRANGE

Re-arrangement (or quantile) in 1d – plots

- Canonical random variable for representing $\mu \in \mathcal{P}(\mathbb{R})$
- Simplest example: $X(x) = \frac{1}{N}\sum_{i=0}^{N-1} a_i 1_{[i/N,(i+1)/N)}(x)$
  - rearrangement on $[0, 1)$: $X^*(x) = \frac{1}{N}\sum_{i=0}^{N-1} a_{(i)} 1_{[i/N,(i+1)/N)}(x)$
  - where $a_{(1)} \leq a_{(2)} \leq ... \leq a_{(N)}$ is the non-decreasing rearrangement of $a_1, \cdots, a_N$
  - to get it on $\mathbb{S}$, use contraction of rate $1/2$ and symmetrize

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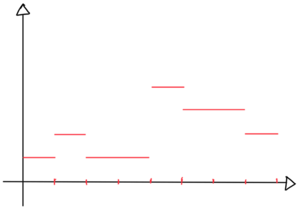

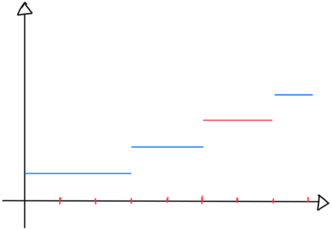

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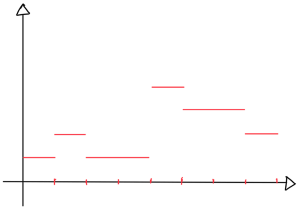

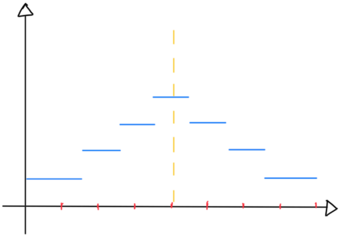

Euler scheme with white noise

- Naive idea (from the general plan)

$$X^h_{n+1} = \left[e^{h\Delta} X^h_n + \int_0^h e^{(h-s)\Delta} dW_{nh+s} \right]^*$$

  - $h > 0$ is a time step

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- Not able to prove *tightness* (i.e., weak compactness)!
- Principle of the analysis taken from Brenier [09]
  - use non-expansion of the re-arrangement

$$\|u^* - v^*\|^2_{2,\mathbb{S}} = \int_{\mathbb{S}} |u^*(x) - v^*(x)|^2 dx \leq \int_{\mathbb{S}} |u(x) - v(x)|^2 dx = \|u - v\|^2_{2,\mathbb{S}}$$

with $u^* = X^h_{n+1}$ and $\underbrace{v^* = e^{((n+1)-N)h\Delta} X^h_N}_{\text{sym. } \nearrow}$ for $N \leq n$

and $u = e^{h\Delta} X^h_n + \int_0^h e^{(h-s)\Delta} dW_{nh+s}$ and $v = v^*$ for $N \leq n$

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- Principle of the analysis taken from Brenier [09]

$$\mathbb{E}\left[\left\| X^h_{n+1} - e^{((n+1)-N)h\Delta} X^h_N \right\|^2_{2,\mathbb{S}} \right]$$
$$\leq \mathbb{E}\left[\left\| e^{h\Delta} \left(X^h_n - e^{(n-N)h\Delta} X^h_N \right) \right\|^2_{2,\mathbb{S}} \right] + \underbrace{\mathbb{E}\left[\left\| \int_0^h e^{(h-s)\Delta} dW_{nh+s} \right\|^2_{2,\mathbb{S}} \right]}_{h^{1-\cdots}}$$

  - use contraction of $e^{h\Delta} \rightsquigarrow h^{-1} h^{1-\cdots} = h^{-\cdots} \rightsquigarrow$ BAD
  - need to combine $e^{h\Delta}$ and $* \rightsquigarrow$ NO SIMPLE WAY

Euler scheme with colored noise

- Replace white noise by colored noise

$$\widetilde{W}_t(x) = \sum_{m\in\mathbb{Z}} m^{-\lambda} W_t^m e_m(x)$$

where $\lambda \in (1/2, 1]$ and $((W_t^m)_{t\geq 0})_{m\in\mathbb{Z}}$ are independent Brownian motions

  - $\mathbb{E}[||\widetilde{W}_t(\cdot)||_2^2] = ct < \infty$
  - the noise takes values in $L^2(\mathbb{S}, \mathrm{Leb})$

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- May wonder why Δ is still needed in the equation
  - for the smoothing effect!! [Da Prato, ...]

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- New scheme

$$X_{n+1}^h = \left[e^{h\Delta} X_n^h + \int_0^h e^{(h-s)\Delta} d\widetilde{W}_{nh+s} \right]^*$$

  - $h > 0$ is a time step
  - get tightness in any $C([0, T]; H^{-1}(\mathbb{S}))$

3. Rearranged SHE

Equation satisfied by limit process

- Brenier's work $\rightsquigarrow$ infinitesimal impact of re-arrangement = reflection on symmetric non-decreasing functions

- Get a reflected SHE

$$dX_t = \Delta X_t dt + d\widetilde{W}_t + d\eta_t$$

  - recall that $X_t \in L^2(\mathbb{S}, \mathrm{Leb})$ by symmetric non-decreasing

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∘ reflected SPDE $\rightsquigarrow$ Donati-Martin & Pardoux, Nualart & Pardoux, Zambotti (reflection to preserve positivity), Barbu & Da Prato & Tubaro, Röckner & Zhu and & Zhu (more general treatment)

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- What is η_t?

$$d\eta_t = \left(e^{dt\Delta}\, X_t + \int_0^{dt} e^{(dt-s)\Delta} d\widetilde{W}_{t+s}\right)^* - \left(e^{dt\Delta}\, X_t + \int_0^{dt} e^{(dt-s)\Delta} d\widetilde{W}_{t+s}\right)$$

  - if u is smooth and symmetric non-decreasing

$$\langle u, d\eta_t \rangle_{2,\mathbb{S}} \geq 0$$

  - if $(z_t)_{t\geq 0}$ is smooth, symmetric $\nearrow$ and varies smoothly in time

$$\int_0^t \langle z_s, d\eta_s \rangle_{2,\mathbb{S}}$$

  makes sense (think of Stieltjes-integral) and ≥ 0

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$$\int_0^t \langle z_s, d\eta_s \rangle_{2,\mathbb{S}} = \sum_m \int_0^t \langle z_s, e_m \rangle_{2,\mathbb{S}} d\langle \eta_s, e_m \rangle_{2,\mathbb{S}}$$

makes sense (think of Stieltjes-integral) and ≥ 0

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- Non-decreasing property of the reflection term

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- Implies uniqueness as in finite dimension

4. Smoothing Effect

Result

- Smoothing effect of the semi-group is standard folklore of SPDEs

$$\mathcal{P}_t : X_0 \in L^2(\mathbb{S}, \mathrm{Leb}) \mapsto \mathbb{E}\left[\varphi\left(X_t^{X_0^*}\right)\right]$$

  - for $\varphi : L^2(\mathbb{S}, \mathrm{Leb}) \to \mathbb{R}$ (or $\mathcal{P}^2(\mathbb{R}) \to \mathbb{R}$) bounded and measurable

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- Bound on the Lipschitz constant

$$\left|\mathcal{P}_t\varphi((X_0 + z)^*) - \mathcal{P}_t\varphi(X_0^*)\right| \leq \frac{C_T}{t^{(1+\lambda)/2}} \|\varphi\|_\infty \|z\|_{L^2}$$

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- Discussion on the rate
  - blow-up exponent $(1 + \lambda)/2 \in (3/4, 1)$, close to $3/4$ for $\lambda \sim 1/2$
  - NOT AS GOOD as in finite dimension (blow up like $t^{-1/2}$)
  - but INTEGRABLE in small time, crucial for nonlinear models

5. Combining with idiosyncratic noise

Prospects

- Applications

  ◦ drifted equations and related gradient descent ✓

$$dX_t(x) = b\big(X_t(x), \mathrm{Leb}_{\mathbb{S}} \circ X_t^{-1}\big)dt + \Delta X_t(x)dt + d\widetilde{W}_t(x) + d\eta_t(x)$$

  ◦ nonlinear models (in the sense of BSDEs, or equivalently, nonlinear PDEs on $\mathcal{P}(\mathbb{R})$ ✓

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- Extensions
  - higher rearrangement given by optimal transport ✗
  - copula ?
  - below: (idiosyncratic vs. common) noise for

$$\partial_t \mu_t = -\mathrm{div}(b_t(\cdot, \mu_t)\mu_t) + \frac{1}{2}\Delta\mu_t$$

not reachable by Dello Schiavo or Sturm constructions

Principle

- Consider $(B_t)_{t\geq 0}$ another Brownian motion constructed on some Ω (corresponding to idiosyncratic noise), whilst $\widetilde{W}$ (corresponding to common noise) is constructed on some Ω^0
  - naively, consider (something like)

$$X_t(x, \omega_0) + B_t(\omega)$$

but DOES NOT make sense because each 'x' should have its own noise

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$$\mu_t(\omega_0) = \mathcal{L}_{x,\omega}\Big(X_t(x, \omega_0) + B_t(\omega)\Big) = g_t \star \mathcal{L}_x\Big(X_t(x, \omega_0)\Big)$$

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- Make it proper: Trotter-Kato

$$X_0 \rightsquigarrow X_{dt}^{X_0}(x, \omega_0) \rightsquigarrow g_{dt} \star \mathcal{L}_x\Big(X_{dt}^{X_0}(x, \omega_0)\Big)$$

New scheme

- Subdivision $t_0 = 0 < t_1 = h < t_2 = 2h < \cdots < t_n = nh$,

$$X^h_{t_n} \rightsquigarrow \mathrm{RSHE}(X^h_{t_n}, [t_n, t_n + 1]) \rightsquigarrow \star g_h$$

  ◦ on a time step $[t_n, t_{n+1}] \rightsquigarrow$ RSHE dynamics

$$\widetilde{X}^h_t(x) = X^h_{t_n}(x) + \int_{t_n}^{t} \Delta \widetilde{X}^h_s(x) ds + (\widetilde{W}_t - \widetilde{W}_{t_n})(x) + (\eta^h_t - \eta^h_{t_n})(x)$$

for $t \in [t_n, t_n + 1]$

  ◦ at time t_{n+1}, convolution

$$X^h_{t_{n+1}}(x) = \left(\mathrm{Leb}_{\mathbb{S}} \circ (\widetilde{X}^h_{t_{n+1}})^{-1} \star g_h\right)^{-1}(x), \quad x \in \mathbb{S}$$

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- New term to understand is

$$S^h_{t_n} := \sum_{j=1}^n \left[X^h_{t_j} - \widetilde{X}^h_{t_j}\right]$$

Approximating the new term

- Standard computation
  - μ a probability measure, with F_μ as cdf and F_μ^{-1} as quantile on $\mathbb{S}$
  - ϕ symmetric (non-decreasing)

$$\int_{\mathbb{S}} \phi(x)(F^{-1}_{\mu\star g_h}(x) - F_\mu^{-1}(x))dx = \frac{1}{2}\int_0^h \int_{\mathbb{S}} \frac{\phi'(x)}{(F_\mu \star g_r)^{-1}}(x)dx\,dr$$

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  - justify existence of term above

$$\frac{1}{2}\mathbb{E}\int_0^h \int_{\mathbb{S}} \frac{\phi'(x)}{(\mathrm{Leb}_{\mathbb{S}} \circ (X^h_{t_j})^{-1} \star g_r)^{-1}(x)}dx\,dr \le C(\|\phi'\|_\infty)$$

Retrieving the derivative of the entropy

- Limiting equation

$$dX_t(x) = -\Big(\frac{1}{X_t'(x)}\Big)' dt + X_t''(x)dt + d\widetilde{W}_t(x) + d\eta_t(x)$$

with $\frac{1}{2}\mathbb{E}\int_0^T \int_{\mathbb{S}} \frac{1}{|X_t'|}(x)dx\,dt \leq C$

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- Interpretation
  - new term corresponds to derivative of entropy

$$\frac{d}{d\epsilon}_{|\epsilon=0}\Big[-\int_{\mathbb{S}} \ln(|X'(x)+\varepsilon\phi'(x)|)dx\Big] = -\int_{\mathbb{S}} \frac{\phi'(x)}{X'(x)}dx = \int_{\mathbb{S}} \phi(x)\Big(\frac{1}{X'(x)}\Big)' dx$$

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- More
  - uniqueness is ✓
  - need for reflection? regularization properties?