Noise sensitivity for the 2D Stochastic Heat Equation and directed polymers

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University of Milano-Bicocca

Workshop on "Workshop on Irregular Stochastic Analysis "

INdAM Meeting at Palazzone di Cortona $\sim$ 24 June 2025

Collaborators

Joint work with

Anna Donadini

Collaborators

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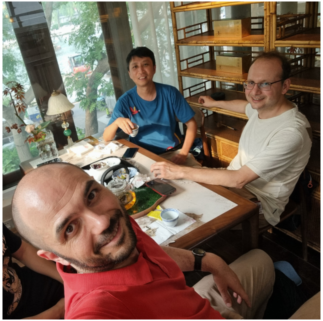
Anna Donadini

Previous works with

Rongfeng Sun (NUS)

Nikos Zygouras (Warwick)



Outline

1. The critical 2D SHF

2. Which equation for the SHF?

3. Noise sensitivity

The Stochastic Heat Equation

Heat equation with multiplicative singular potential $t \geq 0,\ x \in \mathbb{R}^d$

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

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Ito-Walsh / Robust solution theories

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Natural candidate solution: the critical 2D Stochastic Heat Flow (SHF)

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Convergence of $u_N(t,\varphi) = \int_{\mathbb{R}^2} u_N(t,x)\,\varphi(x)\,\mathrm{d}x$ as $N \to \infty$?

Renormalisation

Mean convergence $$\mathbb{E}\big[u_N(t,\varphi)\big] \xrightarrow[N\to\infty]{} \int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x$$ (easy)

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Convergence in law of $u_N(t,\varphi)$? $\iff$ of the measure $u_N(t,x)\,\mathrm{d}x$?

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$\mathscr{U}^{\vartheta}$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of random measures on $\mathbb{R}^2$

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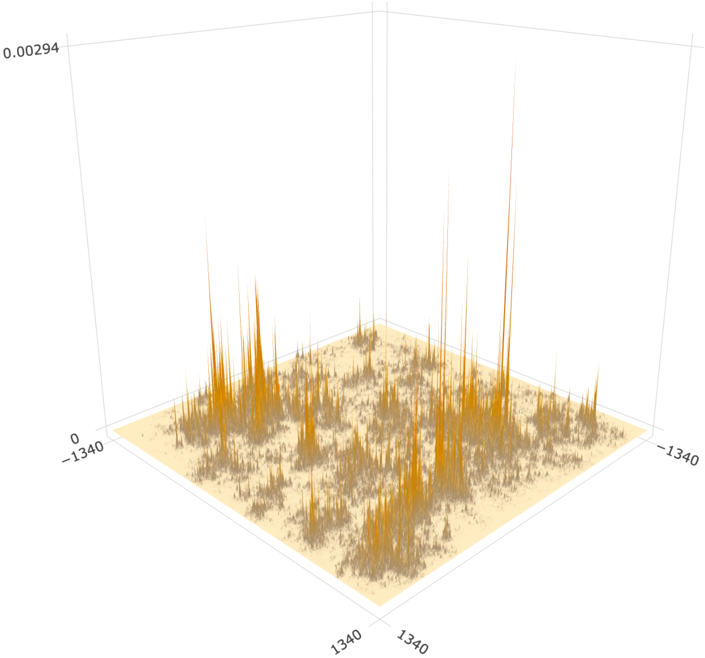
- Universality w.r.t. approximation scheme [C.S.Z. 23] [Tsai 24+]



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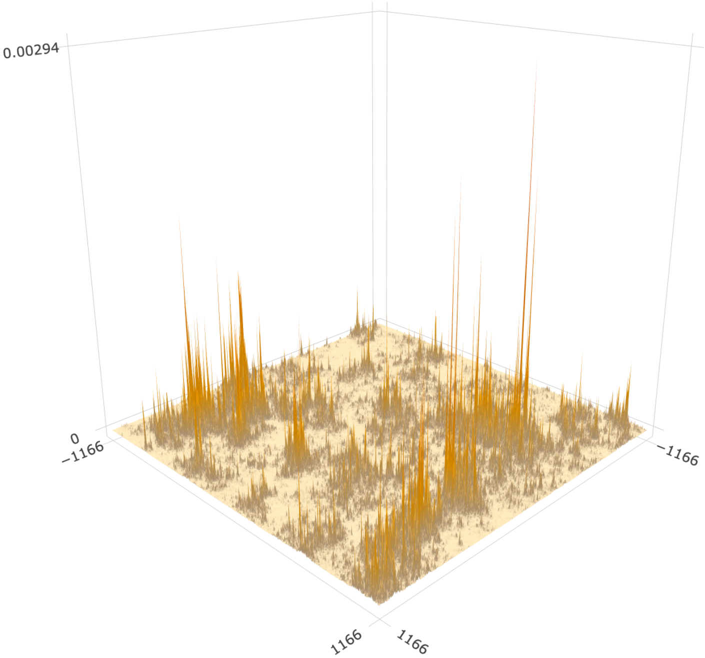
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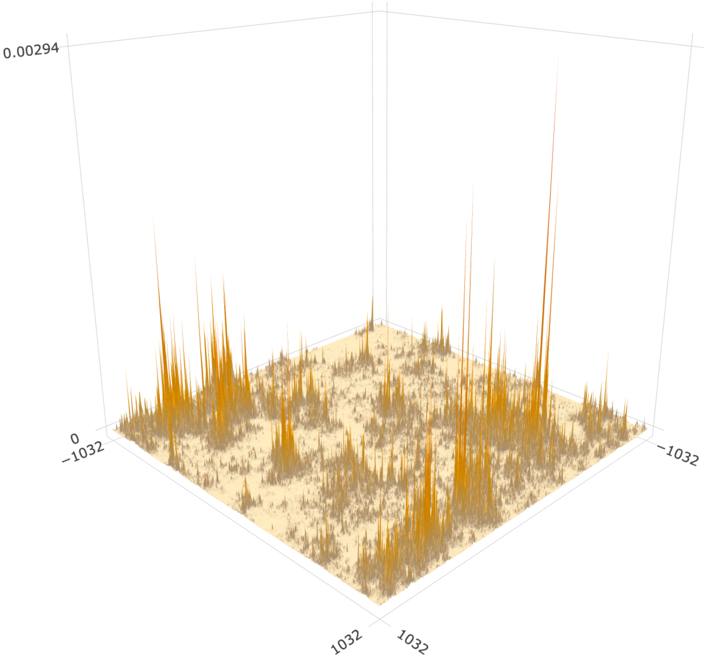

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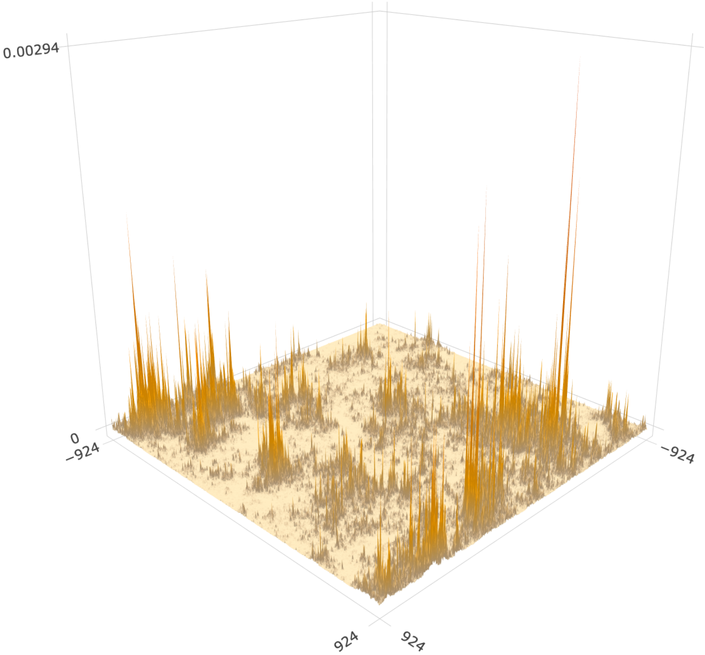
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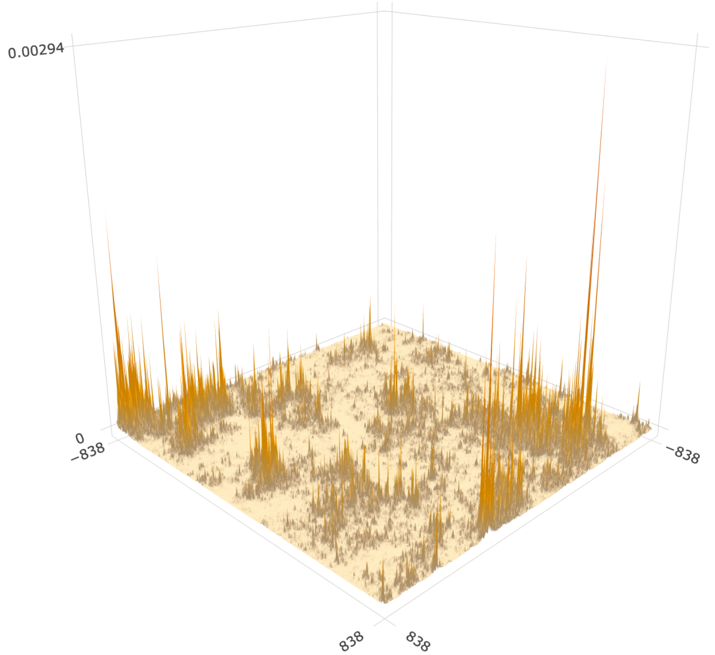

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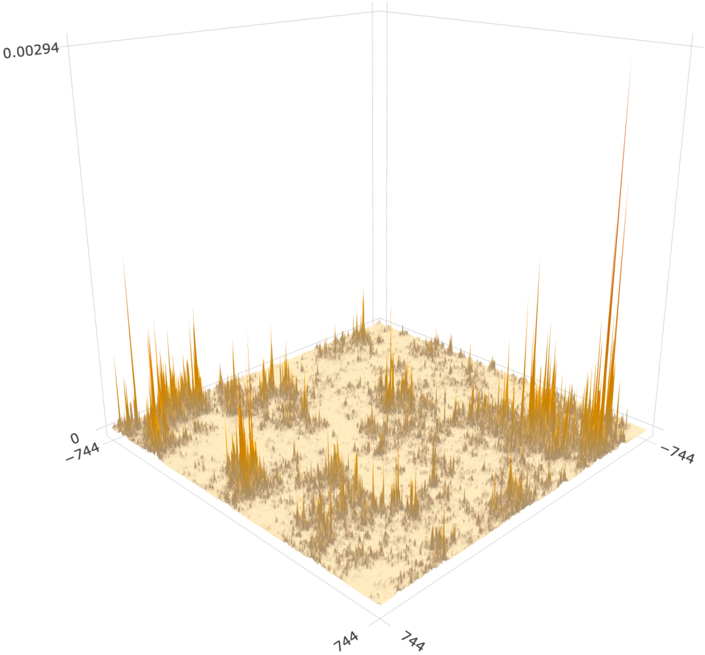

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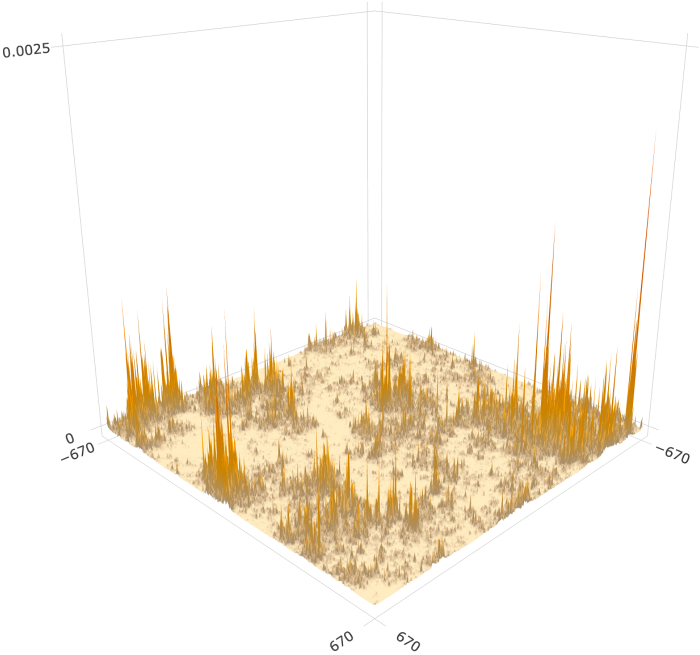
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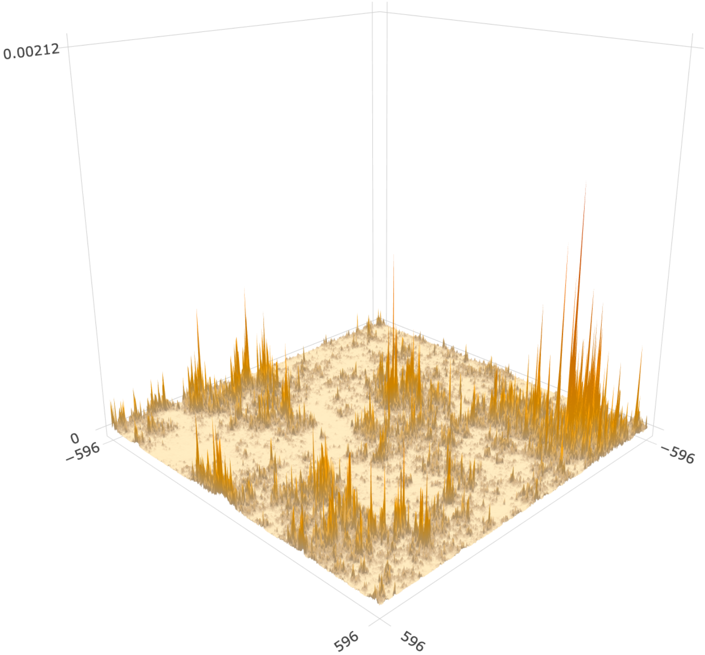
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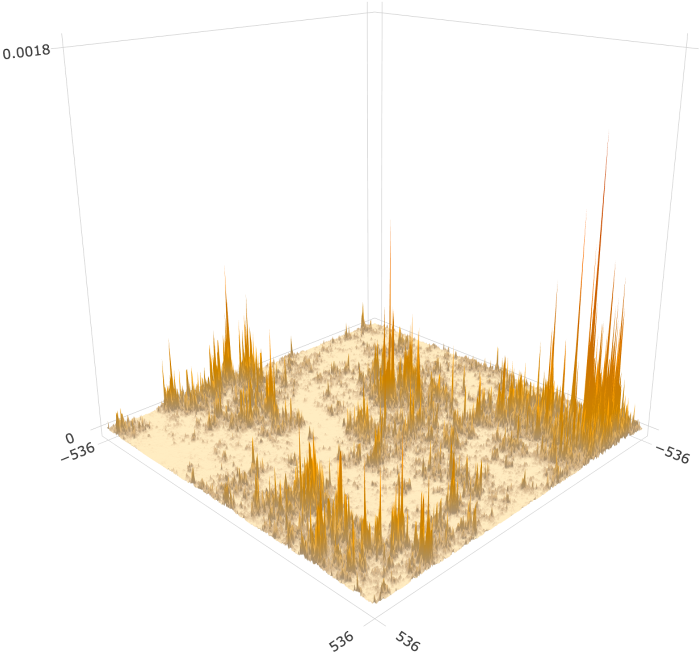
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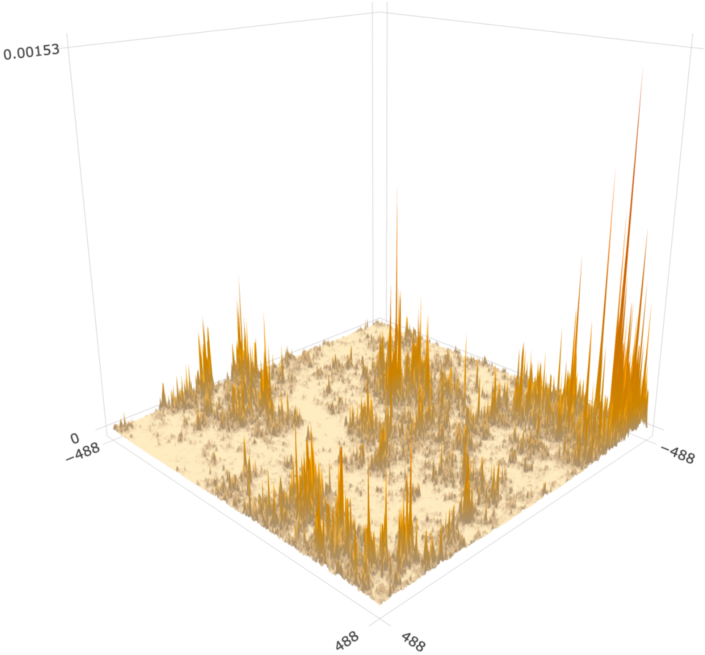
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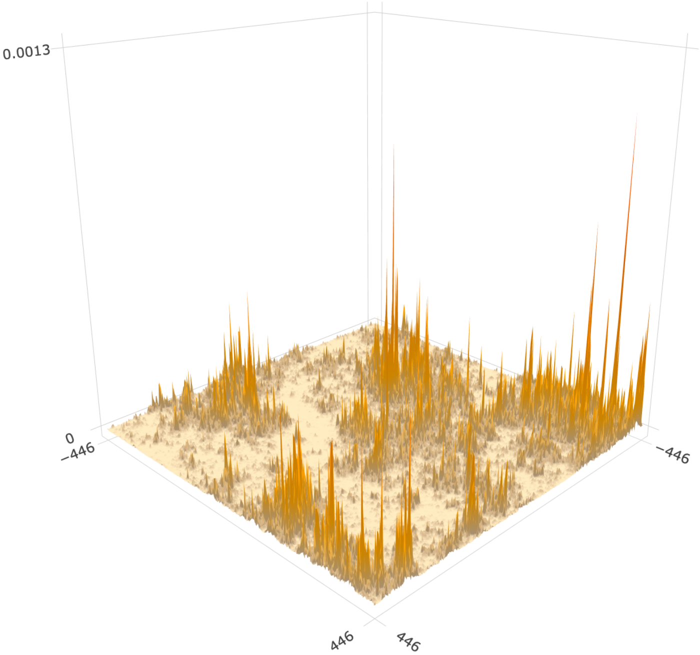
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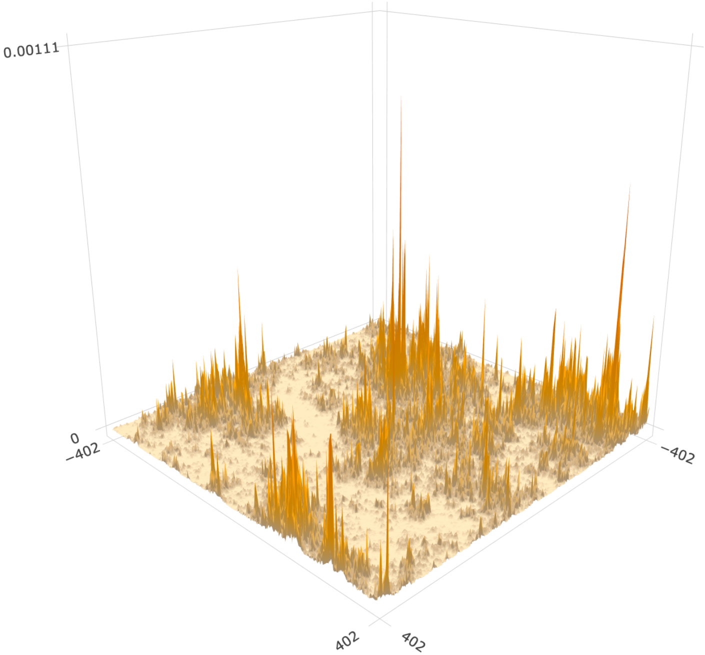

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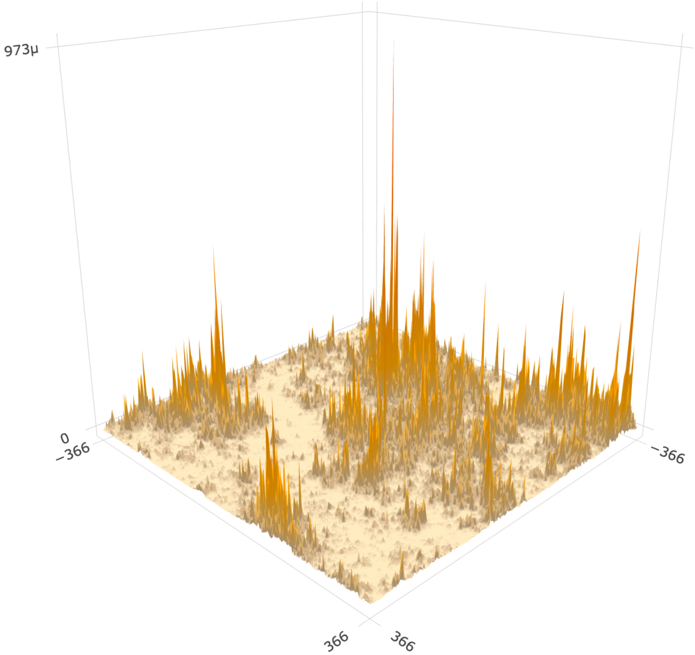

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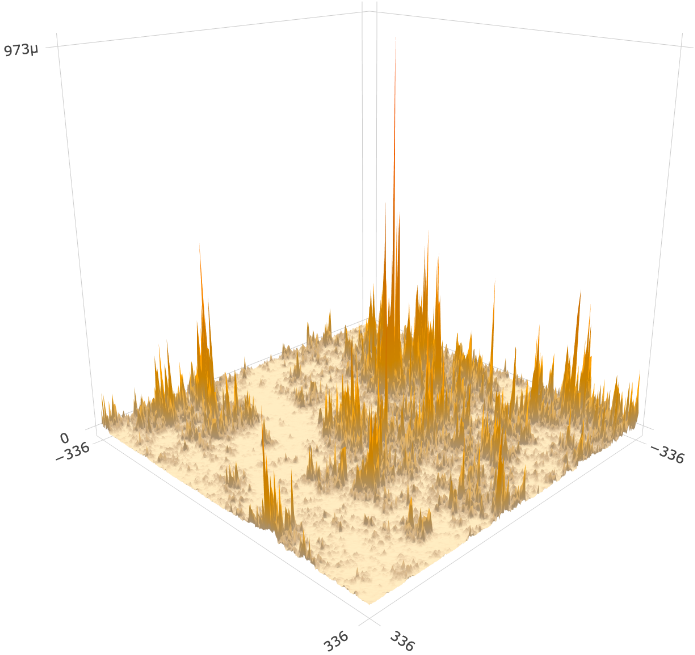

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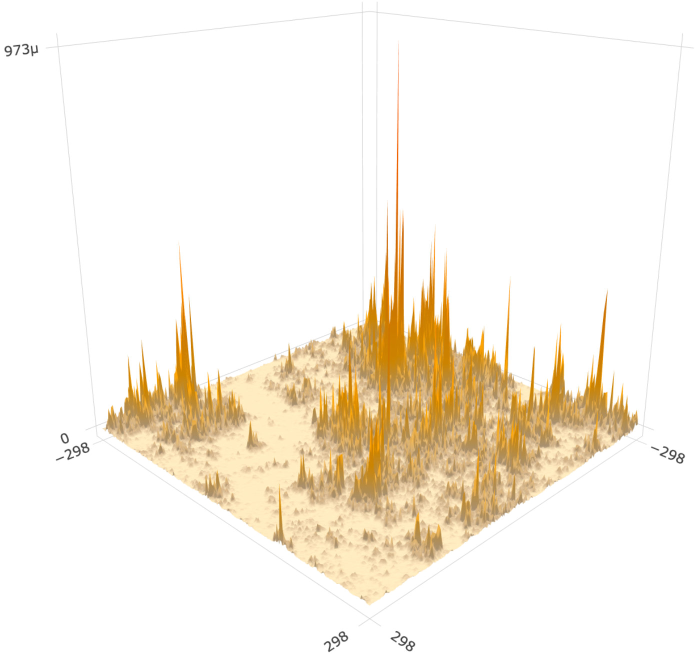
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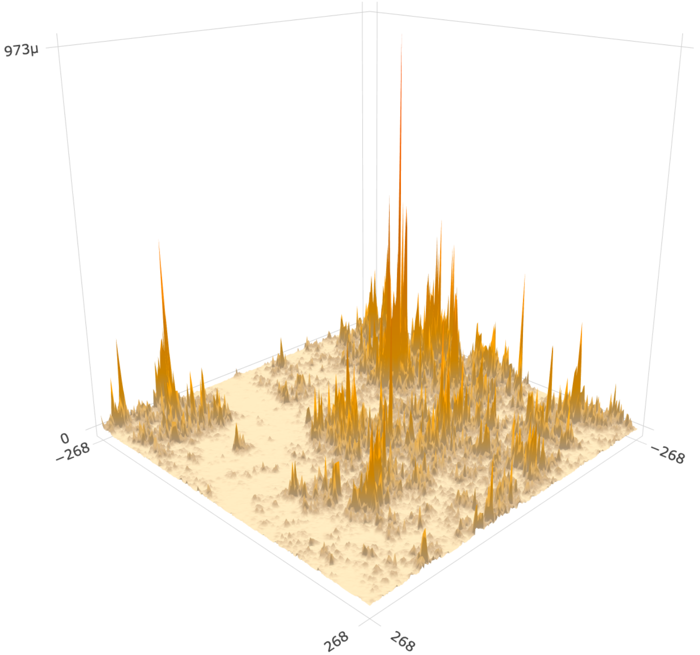

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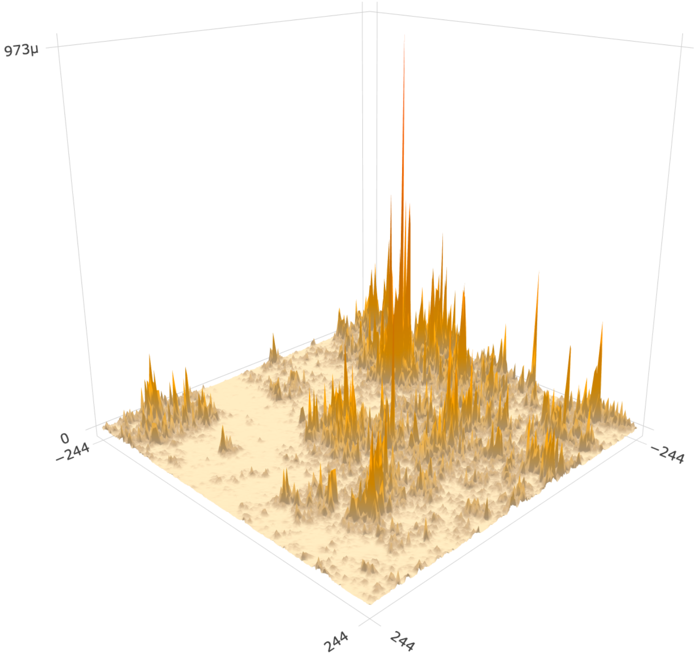

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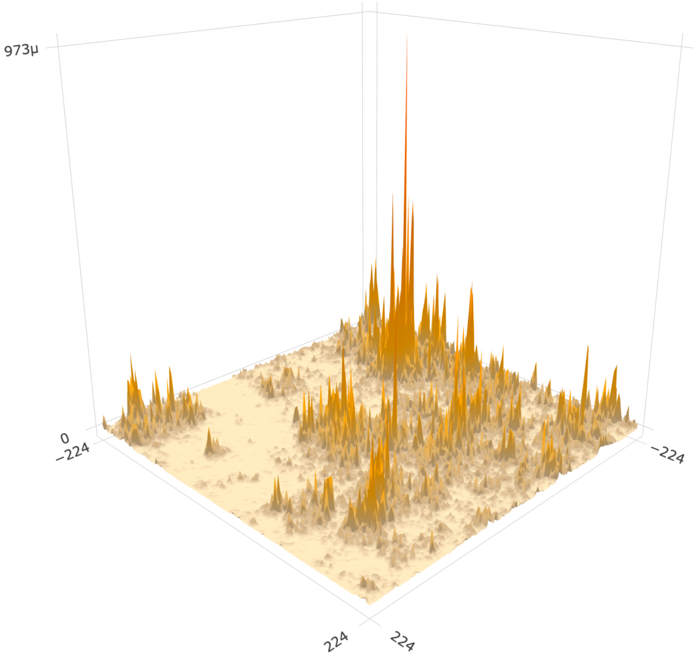
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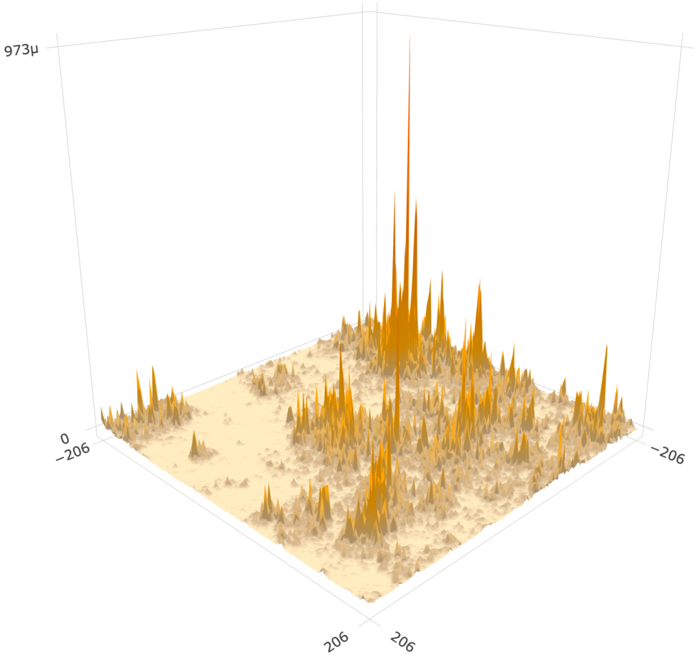
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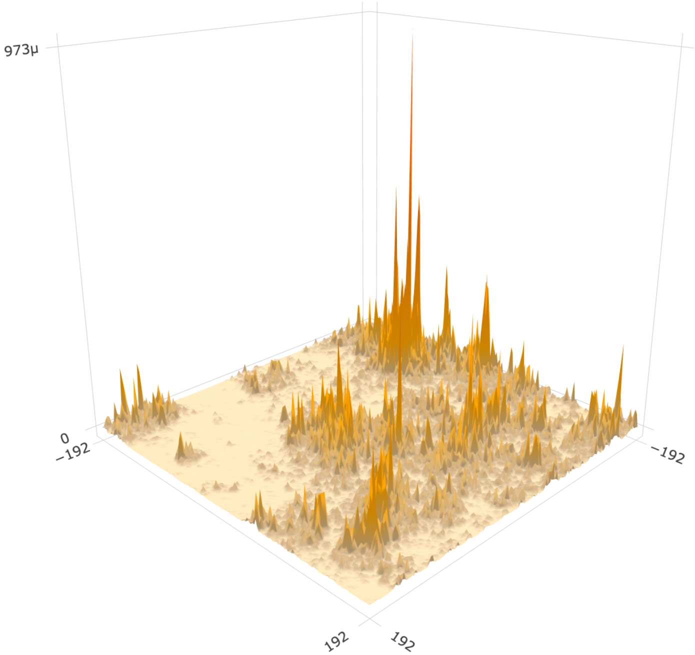
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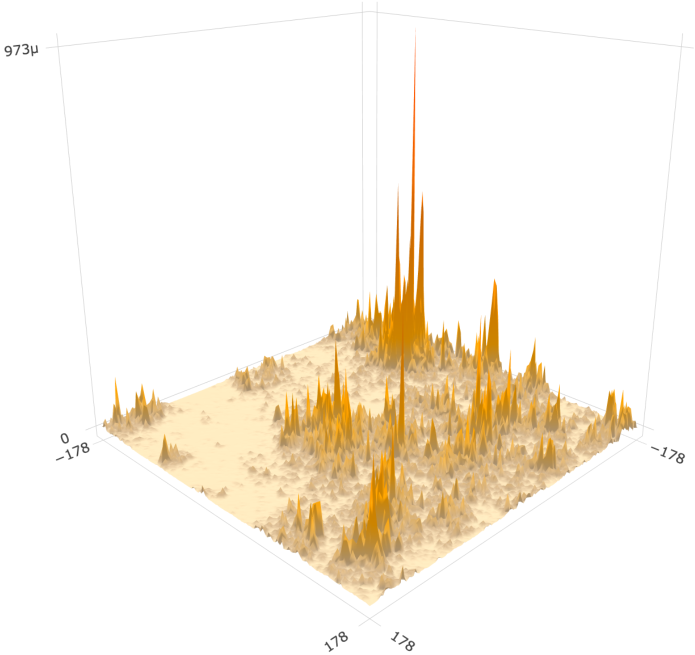

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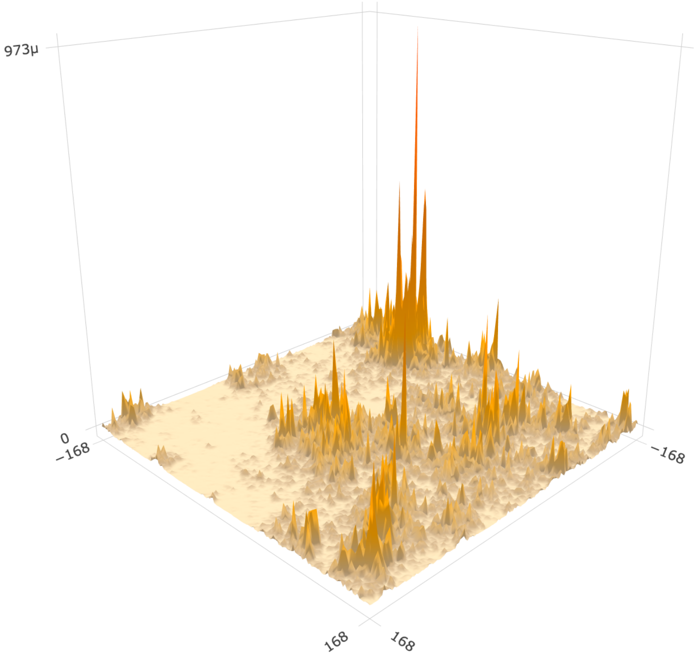
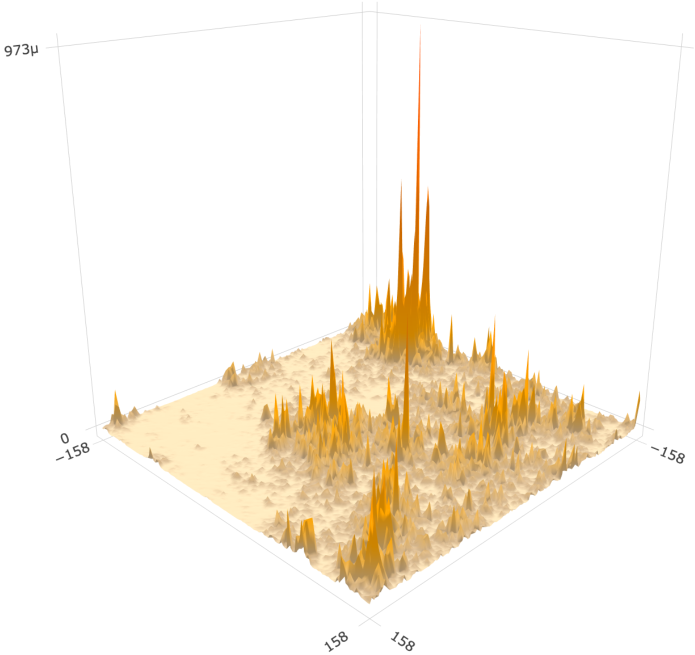


973µ
0
−168
168
168
−168


973µ
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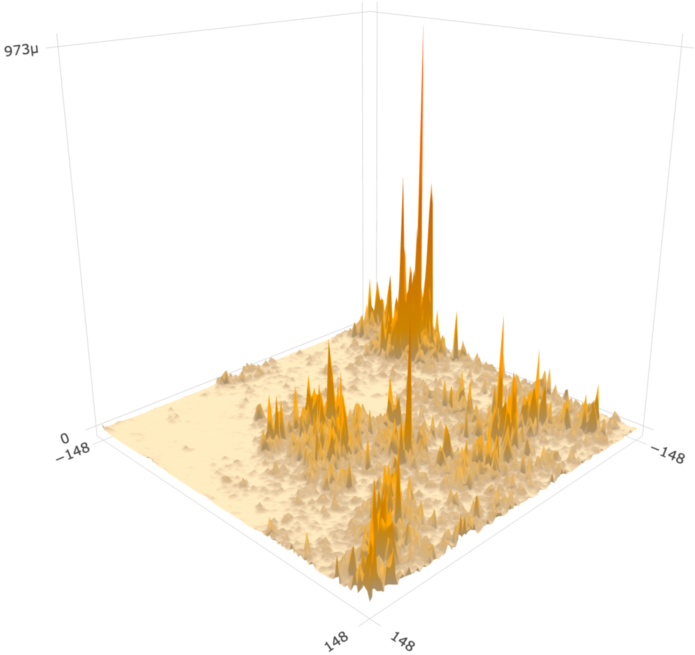
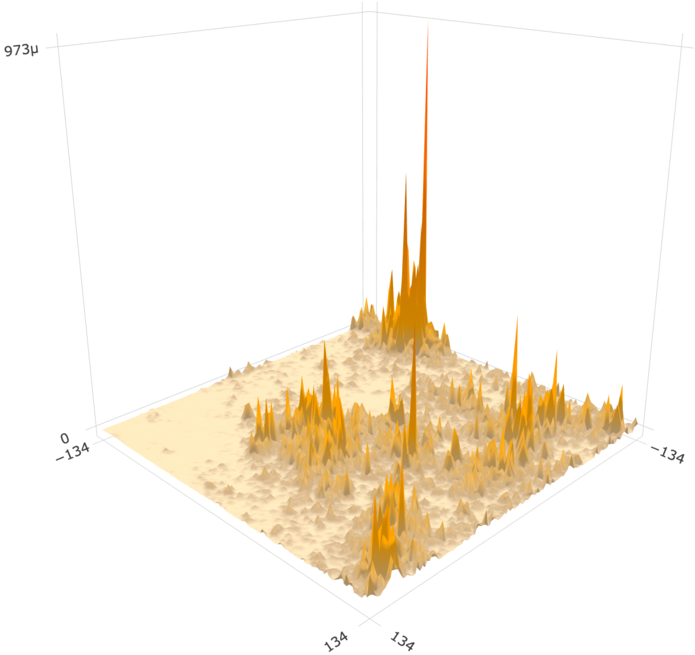
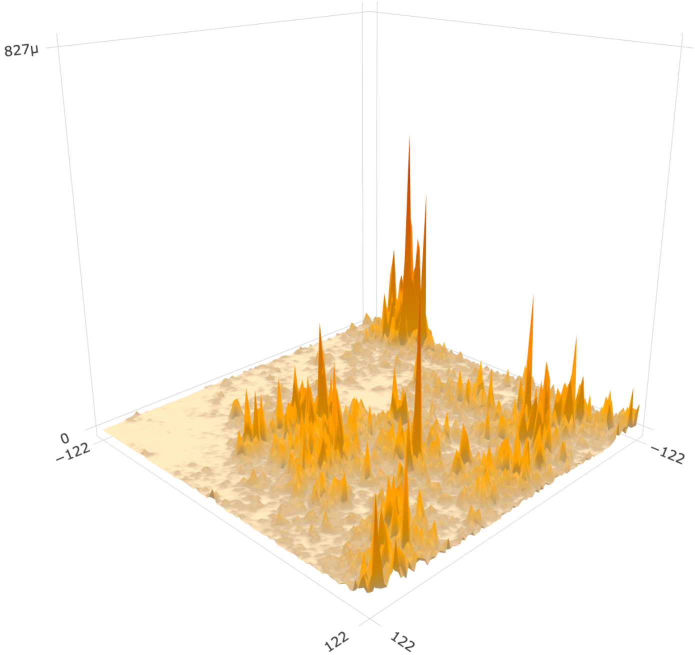
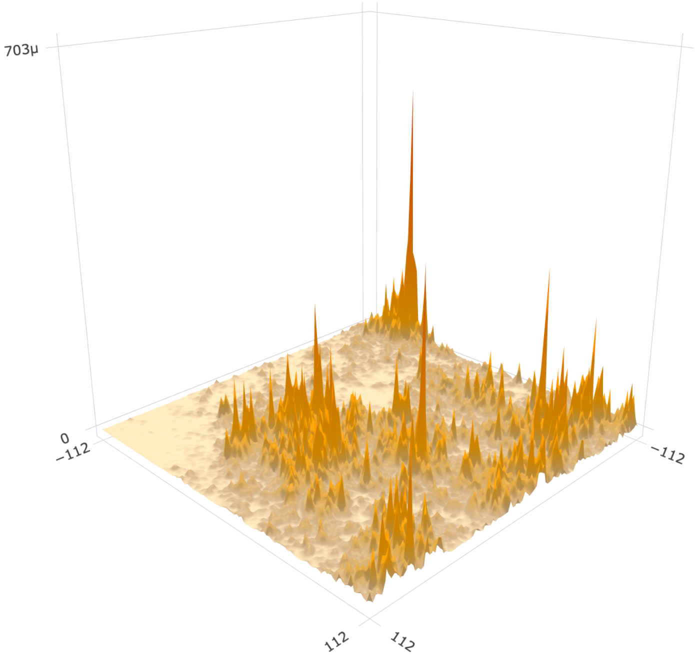
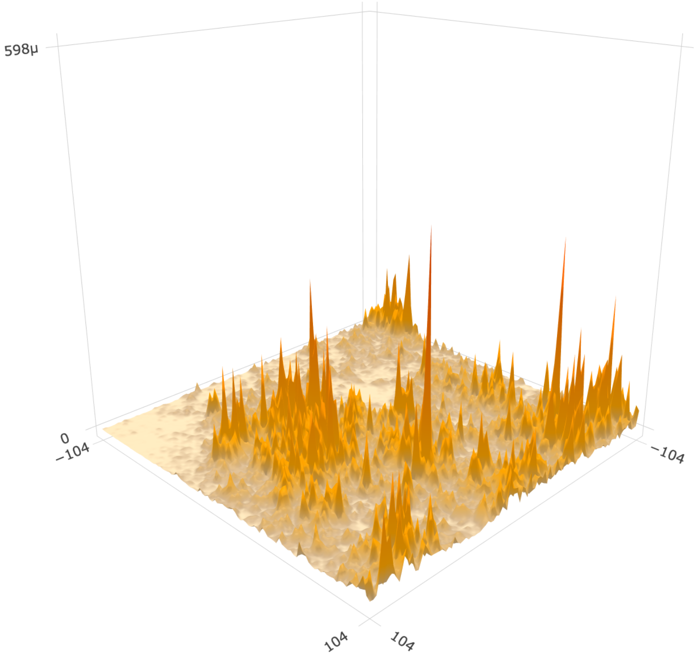
−158
158
158
−158

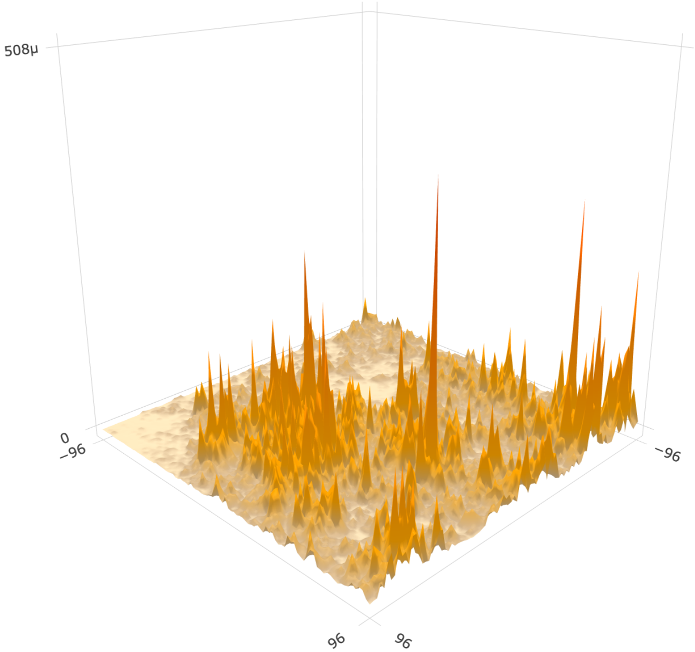

973μ
0
−148
−148
148
148

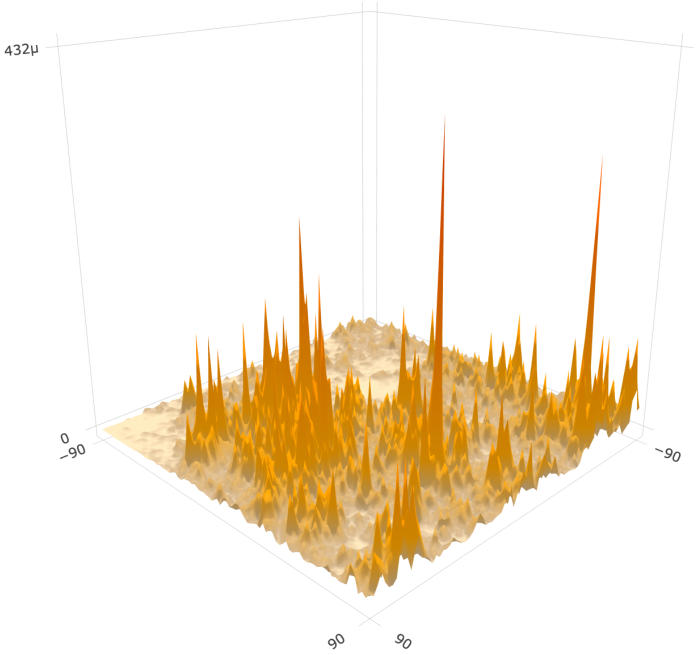

973μ
0
−134
134
134
−134


827μ
0
−122
122
122
−122


703μ
0
−112
−112
112
112


598μ
0
−104
104
104
−104


508μ
0
−96
96
96
−96


432µ
0
−90
90
90
−90

Outline

1. The critical 2D SHF
2. Which equation for the SHF?
3. Noise sensitivity

SHF and the white noise

Does SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ satisfy a SPDE driven by white noise $\xi(t, x)$?

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$$\langle \xi, \psi \rangle = \lim_{N\to\infty} \int \xi_N(t,x)\, \psi(t,x)\, \mathrm{d}x \qquad \text{in distribution} \qquad \psi \in C_c^\infty(\mathbb{R}^{1+2})$$

No equation for the SHF

Theorem [C.–Donadini 25+]

$$\left(\xi_N, u_N\right) \xrightarrow[N\to\infty]{d} \left(\xi, \mathscr{U}^{\vartheta}\right)$$

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Recently proved: $\mathscr{U}^{\vartheta}$ is a "black noise" (à la Tsirelson) [Gu-Tsai arXiv 25]

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(partition function of 2D directed polymer in random environment)

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Define "ε-perturbation" $\omega^\varepsilon = (\omega_i^\varepsilon)_{i=1,2,\dots}$ $\omega_i^\varepsilon := \begin{cases} \omega_i & \text{w. prob. } 1-\varepsilon \\ \widetilde{\omega}_i \perp\!\!\!\perp \omega_i & \text{w. prob. } \varepsilon \end{cases}$

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We call $(f_N)_{N\in\mathbb{N}}$ noise sensitive if [Garban–Steif 14] [O'Donnel 14]

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$\iff \quad \phi\left(f_N(\omega^\varepsilon)\right)$ is noise sensitive $\qquad \forall \phi \in C_c^\infty(\mathbb{R})$

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Spectral criterion

Noise sensitivity $\iff$ $\forall d \in \mathbb{N}$: $\| f_N^{(d)} \|_2^2 \xrightarrow[N\to\infty]{} 0$

The BKS Theorem

Boolean setting: binary functions $f(\omega)$ of binary variables ω_i

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$\forall \varepsilon > 0$: $\mathrm{Cov}\left[f(\omega^\varepsilon), f(\omega)\right] \leq C\, \mathscr{W}(f)^{\alpha\,\varepsilon}$ [Keller–Kindler 13]

Influences beyond the Boolean setting

Define $\delta_i f := f - \mathbb{E}_i[f]$ with $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\sigma(\omega_j : j \neq i)]$ [Talagrand 94]

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It is the L^1 influence that is relevant for us: $\mathscr{W}(f) := \sum_i I_i^{(1)}(f)^2$

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Back to SHE

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$\phi(u_N(t,\varphi))$ is noise sensitive $\forall$ Lipschitz ϕ if the $\boldsymbol{\omega}_i$'s take finitely many values

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Our proof refines Keller-Kindler: optimal estimate for binary ω_i's

$$\mathbb{C}\mathrm{ov}\left[f(\omega^{\varepsilon}), f(\omega)\right] \leq \mathscr{W}(f)^{\frac{\varepsilon}{2-\varepsilon}+o(1)}$$

Conclusion

We extended the BKS Theorem beyond the Boolean setting

- Robust conditions for noise sensitivity (stable under composition)
- Quantitative bounds

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$$\mathbb{C}\mathrm{ov}\left[f(\omega^{\varepsilon}), f(\omega)\right] \leq \mathscr{W}(f)^{\frac{\varepsilon}{2-\varepsilon}+o(1)}$$

The assumption that ω_i's take finitely many values can hopefully be removed

Grazie

Directed Polymer in Random Environment

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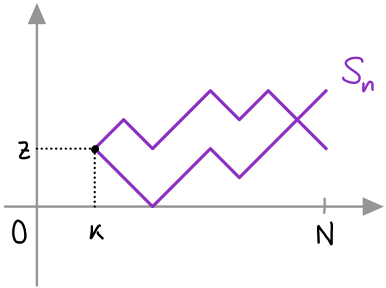

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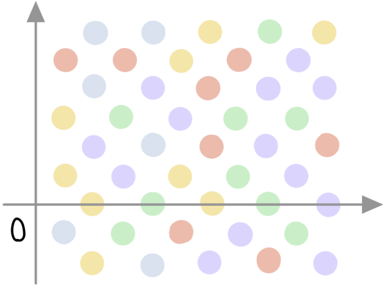

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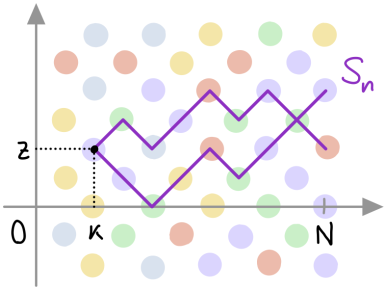

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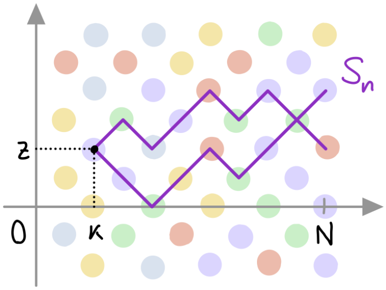

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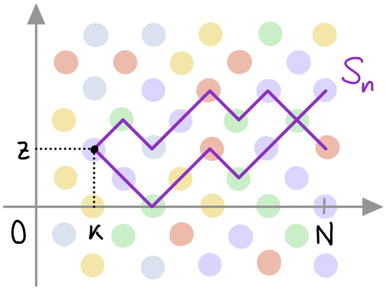

Partition Functions
$(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

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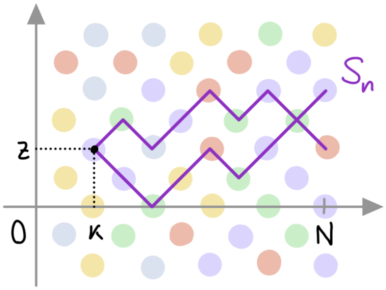

Partition Functions

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$$Z_{N,\beta}^{\omega}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Partition functions and SHE

Diff. rescaled partition functions

$$Z^{\omega}_{N,\beta}\big(N(1-t), \sqrt{N}x\big) \qquad \text{(time rev.)}$$

Partition functions and SHE

Diff. rescaled partition functions $=$ discretized SHE solutions

$$Z^{\omega}_{N,\beta}\big(N(1-t),\sqrt{N}x\big) \;=\; u_N(t,x) \qquad \text{(time rev.)}$$

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Diff. rescaled partition functions = discretized SHE solutions

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Partition functions solve a difference equation: with $\xi_N \approx \omega$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}\; u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(reg-SHE)}$$

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Discrete analogue of Feynman-Kac

$$u_N(t,x) \approx \mathrm{E}\left[\mathrm{e}^{\beta_{\mathrm{SHE}} \int_{1-t}^{1} \xi(s,B_s) - \frac{1}{2}\beta_{\mathrm{SHE}}^2 t} \,\middle|\, B_{1-t} = x \right]$$