Point vortices and their links to PDEs

Franco Flandoli, Scuola Normale Superiore

Irregular Stochastic Analysis

Cortona

General remarks of Mean Field problems

Consider an interacting particle system of the Mean Field type

$$dX_t^i = \frac{1}{N}\sum_{j=1}^{N} K\left(X_t^i, X_t^j\right) dt + \sigma dW_t^i$$

and the PDE

$$\partial_t u(x,t) + \operatorname{div}\left(u(x,t)\int K(x,y)\,u(y,t)\,dy\right) = \frac{\sigma^2}{2}\Delta u$$

satisfied (in a weak sense) by the weak limit of the empirical measure

$$\mu_t^N(dx) = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}(dx).$$

There are several approaches to convergence.

General keywords

and references in the case of 2D Euler equations

- Sznitman coupling argument; McKean-Vlasov
- Entropy + Fisher Information arguments ($\sigma \neq 0$, e.g. Fournier-Hauray-Mischler JEMS '14)
- semigroup approach ($\sigma \neq 0$, F.-Olivera-Simon SIMA '20)
- compactness argument (also for $\sigma = 0$; Schochet CPAM '96)
- **PDE comparison** (also for $\sigma = 0$, main topic of this talk)
  - related to the modulated energy approach, Duerinckx SIMA '16, Serfaty Duke '20, Rosenzweig ARMA '22

The **PDE comparison** started probably with Dobrushin FAA '79: the empirical measure of

$$\frac{dX_t^i}{dt} = \frac{1}{N}\sum_{j=1}^{N} K\left(X_t^i, X_t^j\right) dt$$

is already a measure-valued solution of the limit PDE

$$\partial_t u(x,t) + \operatorname{div}\left(u(x,t)\int K(x,y)\,u(y,t)\,dy\right) = 0.$$

Convergence of $\mu_t^N(dx)$ to $u(x,t)\,dx$ becomes then a question of *stability and uniqueness* for the PDE in the space of measures. Handled by Wasserstein metric.

It can be extended to $\sigma \neq 0$ by considering the *identity* satisfied by the empirical measure

$$d\mu_t^N + \operatorname{div}\left(\mu_t^N \int K(x,y)\,\mu_t^N(dy)\right) = dM_t^N$$

Remark: as for Sznitman coupling argument, the PDE comparison provides *quantitative estimates*.

Two important ideas enriched later Dobrushin approach:

1. the use of negative-order topologies instead of Wasserstein metric on measures
2. the application of weak-strong stability arguments, when possible.

The idea of 1 is that, given the measure $\mu_t^N(dx) = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}(dx)$, we may introduce functions like

$$\phi^N(x,t) = \left(\Delta^{-1}\mu_t^N\right)(x) = \frac{1}{N}\sum_{i=1}^N G\left(x, X_t^i\right)$$

and compare

$$\left\|\phi^N(t) - \phi(t)\right\|_X$$

in some function space X, where $\phi(x,t) = \left(\Delta^{-1}u_t\right)(x)$.
Method 2 applies when the equation for $\phi(t)$ has suitable cancellations.
Let us explain these two ideas in the case of the 2D Euler equations.

The 2D Euler equations

An ideal incompressible fluid in a domanin $D \subset \mathbb{R}^2$ is described by the velocity and the pressure fields $u : D \to \mathbb{R}^2$, $p : D \to \mathbb{R}$ satisfying

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

or equivalently by the vorticity field

$$\omega = \nabla^{\perp} \cdot u = -\partial_2 u_1 + \partial_1 u_2$$

satisfying

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

where u is recovered by ω solving

$$\Delta \phi = \omega, \qquad \phi(x,t) = \int_D G(x,y)\, \omega(y,t)\, dy \qquad G(x) \text{ Green fct}$$

$$u(x,t) = \nabla^{\perp} \phi(x,t) = \int_D \nabla^{\perp} G(x,y)\, \omega(y,t)\, dy.$$

with appropriate decay or boundary conditions.

Two basic results on the 2D Euler equations

- Yudovich CMMP '63: in the class

$$\|\omega\|_\infty < \infty \qquad (\text{"Yudovich class"})$$

  there is existence and uniqueness of weak solutions
  - u is only log-Lipschitz, the flow

$$\partial_t \Phi(t,x) = u_{N,\epsilon}(\Phi(t,x),t), \qquad \Phi(0,x) = x$$

    is only Hölder continuous.
- Kato, ARMA '67: in the class $\omega \in C^\alpha$ there is existence and uniqueness of weak solutions
  - u and the flow are Lipschitz,

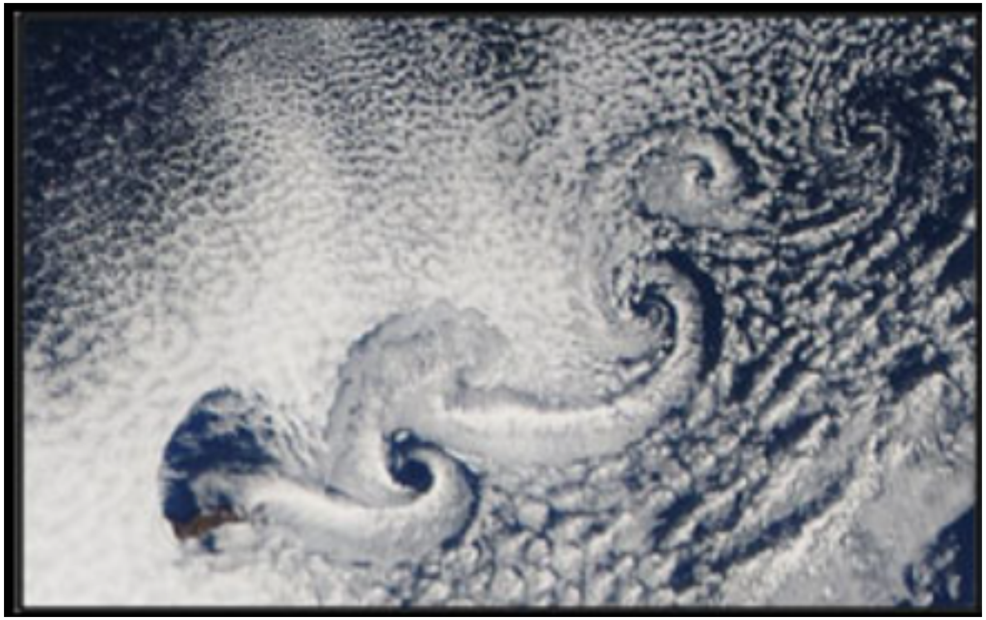
$$\|\nabla u\|_\infty < \infty. \qquad (\text{"Kato class"})$$

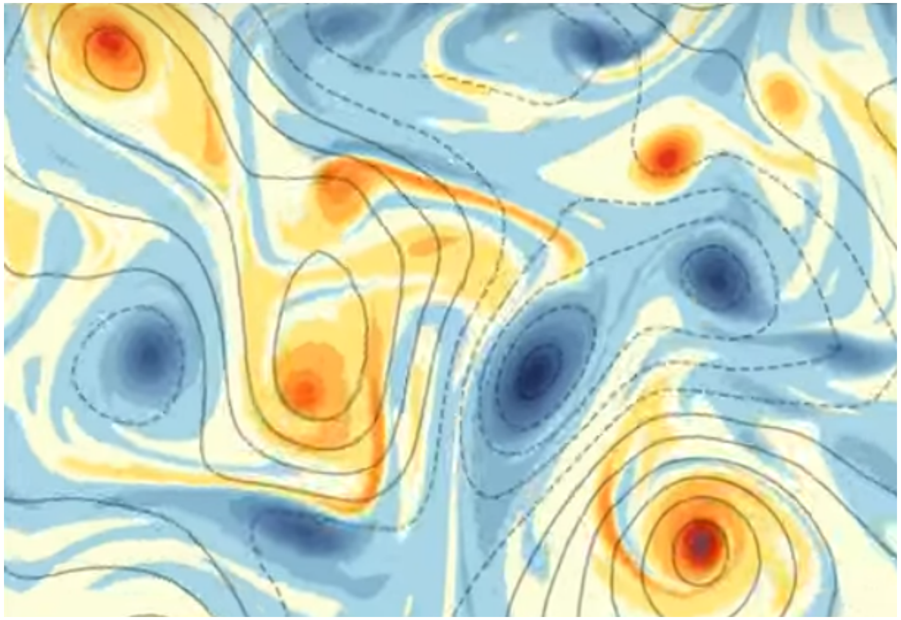
Remark: although $\omega = \nabla^\perp \cdot u$, Yudovich and Kato classes are not the same; ∇u is only BMO when $\|\omega\|_\infty < \infty$.

The velocity field is a complex object, even from the intuitive viewpoint, but the vorticity field is a scalar field, just transported by u

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

hence the intuition about ω is stronger. In particular, the concept of *vortex structure* emerges. There is no precise definition, but we observe "blobs", a "condensates", a "patches" in the profile of ω:





If we have a decomposition at time zero

$$\omega(x,0) = \omega_{blob}(x,0) + \omega_{backgr}(x,0)$$

then we may consider the individual coupled evolutions of $\omega_{blob}(x,t)$ and $\omega_{backgr}(x,t)$

$$\begin{aligned} \partial_t \omega_{blob} + u \cdot \nabla \omega_{blob} &= 0 \\ \partial_t \omega_{backgr} + u \cdot \nabla \omega_{backgr} &= 0 \\ u &= \nabla^{\perp} \Delta^{-1} \left(\omega_{blob} + \omega_{backgr} \right) \\ \omega_{blob}|_{t=0} &= \omega_{blob}(\cdot, 0), \qquad \omega_{backgr}|_{t=0} = \omega_{backgr}(\cdot, 0). \end{aligned}$$

Each substructure is transported by the global velocity field

$$u(x,t) = \int_D \nabla^{\perp} G(x,y) \left(\omega_{blob} + \omega_{backgr} \right)(y,t)\, dy.$$

Evolution of small patches

This coherent-structure vision has been performed on many objects, in particular on small separated patches:

$$\partial_t \omega_\epsilon^i + u_{N,\epsilon} \cdot \nabla \omega_\epsilon^i = 0 \qquad \omega_\epsilon^i (x, 0) = \frac{\Gamma_i}{\pi \epsilon^2} 1_{B\left(x_i^0, \epsilon\right)} (x)$$

$$\text{for each } i = 1, ..., N$$

If the supports $B\left(x_i^0, \epsilon\right)$ at time $t = 0$ are disjoint, they *remain disjoint*:

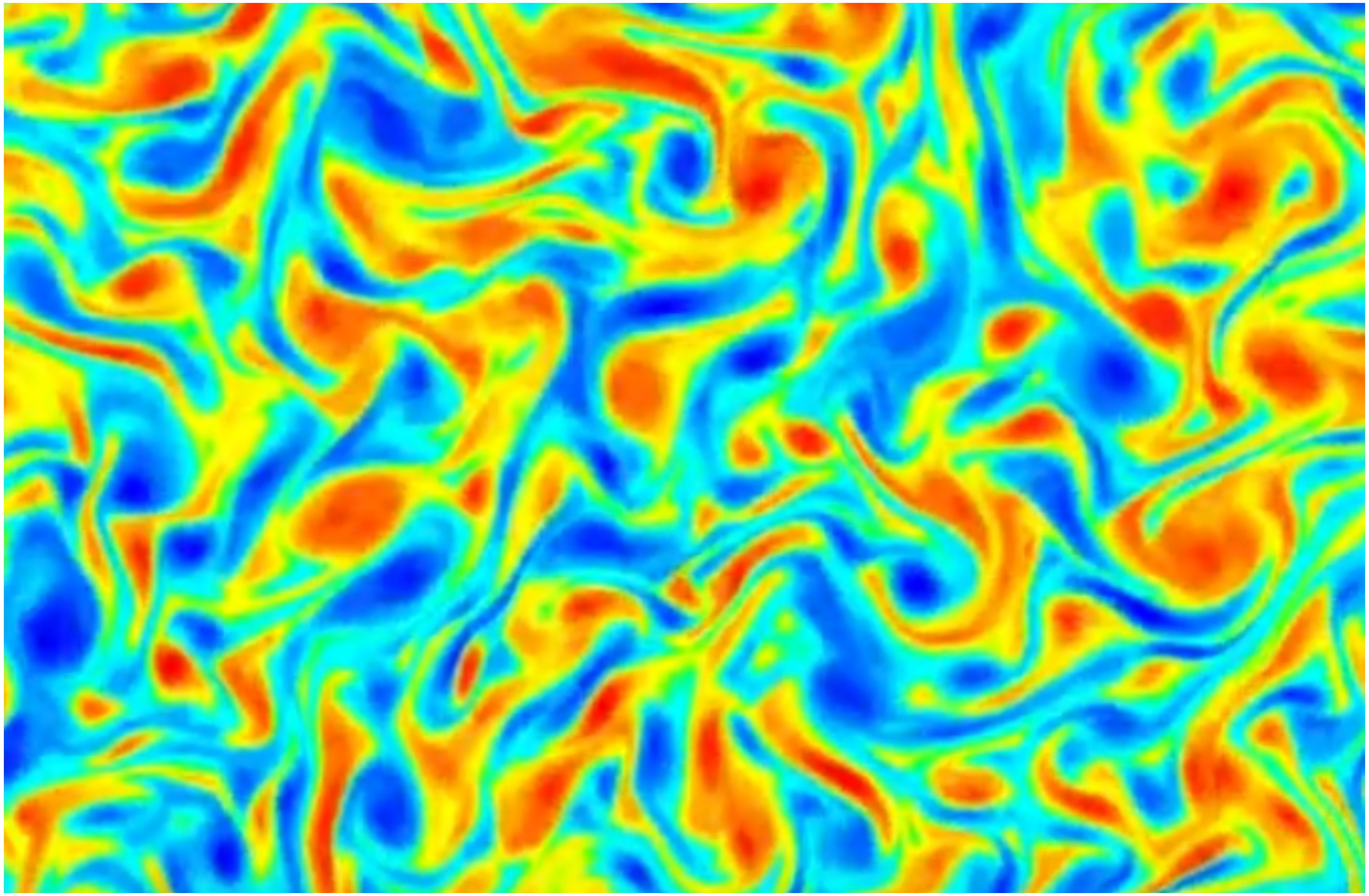
$$\omega_\epsilon^i (x, t) = \frac{\Gamma_i}{\pi \epsilon^2} 1_{S_\epsilon^i(t)} (x)$$

$$\text{where} S_\epsilon^i (t) = \Phi_t \left(B\left(x_i^0, \epsilon\right)\right) \text{ are disjoint sets}$$

$$\partial_t \Phi (t, x) = u_{N,\epsilon} (\Phi (t, x), t), \qquad \Phi (0, x) = x$$

(Hölder flow).

A priori, it may happen that such supports $S_{\epsilon}^{i}(t)$ interlace each other very much:



Theorem (Marchioro and Pulvirenti CMP '93)

Consider the point vortex dynamics

$$\frac{dx_i(t)}{dt} = \sum_{\substack{j=1,\dots,N \\ j\neq i}} \Gamma_j \nabla^{\perp} G\left(x_i(t), x_j(t)\right), \qquad x_i(0) = x_i^0$$

and let $[0,\tau]$ *be an interval where* $(x_i(t))_{i=1,\dots,N}$ *remain at a distance* $r_0 > 0$. *Then, if* ϵ *is small enough w.r.t.* r_0,

$$S_\epsilon^i(t) \subset B\left(x_i(t), r_0\right) \qquad i = 1, \dots, N$$

Moreover, for all $t \in [0,\tau]$,

$$\lim_{\epsilon\to 0} \sum_{i=1}^{N} \omega_\epsilon^i(\cdot, t) = \sum_{i=1}^{N} \Gamma_i \delta_{x_i(t)}.$$

A power law estimate of ϵ w.r.t. r_0 has been given in Flandoli, CPDE '18.

Moreover (e.g. on the torus $\mathbb{T}^2$)

Theorem (Dürr-Pulvirenti CMP '82)

Consider the point vortex dynamics

$$\frac{dx_i(t)}{dt} = \sum_{\substack{j=1,\ldots,N \\ j\neq i}} \Gamma_j \nabla^{\perp} G(x_i(t), x_j(t)), \qquad x_i(0) = x_i^0.$$

For a.e. i.c. $(x_1^0, \ldots, x_N^0)$ *w.r.t. Lebesgue measure on* $\mathbb{T}^{2N}$*, no collapse occurs (for all* $t \geq 0$*).*

Therefore the point vortex dynamics is well defined, without collapse, for a.e. initial condition; and it is approximated by smooth solutions of 2D Euler equations.
In the rest of the talk I will discuss the opposite approximation: every smooth solution of the 2D Euler equations can be approximated by point vortices.

Let us remark that this system fits into the category of *Irregular Stochastic Analysis* if we stress the singularity of the Green function and of the kernel in the point vortex dynamics

$$G(x,y) \sim \frac{1}{2\pi} \log |x-y|, \qquad \nabla^{\perp} G(x,y) \sim \frac{1}{2\pi} \frac{(x-y)^{\perp}}{|x-y|^2}$$

and we stress that we need Probability to study the point vortex dynamics and its convergence to 2D Euler equations, even if the framework is a priori deterministic.
Moreover, extensions and variants to point vortices perturbed by noise have been widely considered and are very important.

The convergence of the empirical measure $\mu_t^N(dx)$ of

$$\frac{dx_i(t)}{dt} = \frac{1}{N} \sum_{\substack{j=1,\dots,N \\ j \neq i}} \nabla^\perp G\left(x_i(t), x_j(t)\right)$$

to the solution $\omega(x,t)\,dx$ of the 2d Euler equations in vorticity form

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

is an example of the general Mean Field problem, since

$$u(x,t) \cdot \nabla \omega(x,t) \overset{\mathrm{div}\, u=0}{=} \mathrm{div}\left(\omega(x,t) \int_D \nabla^\perp G(x,y)\, \omega(y,t)\, dy\right).$$

However, the kernel $\nabla^\perp G$ is very singular

$$\left|\nabla^\perp G(x,y)\right| \leq \frac{C}{|x-y|}$$

and prevents from using classical approaches (for $\sigma = 0$). Apart from the (outstanding) compactness approach performed by Schochet, the first results, by the modulated energy approach, are due to Duerinckx SIMA '16, Serfaty Duke '20, Rosenzweig ARMA '22.

Why negative-order topologies? Instead of investigating the mean field equation for measures $\omega_t(dx)$

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0 \\ u(x,t) &= \int_D \nabla^{\perp} G(x,y)\, \omega_t(dy) \end{aligned}$$

one can investigate the equivalent equation for functions

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

and use, for instance, energy estimates (but see below a difficulty)

$$\left\| u^N(t) - u(t) \right\|_{L^2}^2$$

instead of Wasserstein metric on ω^N, ω.

Moreover, the weak-strong cancellation occurs here (as remarked by Serfaty Duke '20). Assume $u^{(\alpha)}$, $\alpha = 1, 2$ are two smooth solutions (with pressures $p^{(\alpha)}$, $\alpha = 1, 2$) of equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}$$

Set

$$u = u^{(1)} - u^{(2)}, \qquad p = p^{(1)} - p^{(2)}.$$

Then

$$\partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \nabla p = 0$$

(because $u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} = u^{(1)} \cdot \nabla u^{(1)} - u^{(2)} \cdot \nabla u^{(2)}$).

From

$$\partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \nabla p = 0$$

we deduce

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \left\langle u^{(1)} \cdot \nabla u, u \right\rangle + \left\langle u \cdot \nabla u^{(2)}, u \right\rangle + \langle \nabla p, u \rangle = 0$$

but

$$\left\langle u^{(1)} \cdot \nabla u, u \right\rangle = \frac{1}{2}\int_D u^{(1)} \cdot \nabla |u|^2 \, dx = -\frac{1}{2}\int_D \operatorname{div} u^{(1)} |u|^2 \, dx = 0$$

$$\langle \nabla p, u \rangle = \int_D \nabla p \cdot u dx = -\int_D p \operatorname{div} u = 0.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\left\langle u \cdot \nabla u^{(2)}, u \right\rangle.$$

From

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\left\langle u \cdot \nabla u^{(2)}, u \right\rangle$$

we deduce

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \leq \left\|\nabla u^{(2)}\right\|_\infty \|u\|_{L^2}^2$$

hence by Gronwall lemma

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 \, e^{2T\left\|\nabla u^{(2)}\right\|_\infty}.$$

If $u(0) = u^{(1)}(0) - u^{(2)}(0) = 0$, then also $u(t) = u^{(1)}(t) - u^{(2)}(t) = 0$. We have proved the uniqueness of smooth solutions.

Working to minimize the smoothness, we see from

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 \, e^{2T\left\|\nabla u^{(2)}\right\|_\infty}$$

that we need

$$\left\|\nabla u^{(2)}\right\|_\infty < \infty.$$

This is a (nontrivial) regularity condition, however only on one of the two solutions. The weak-strong uniqueness principle is precisely this fact, that we just need regularity of one of the two solutions.
Up to details, one of the most general theorems requires $u^{(1)}(0) = u^{(2)}(0) \in C^{1,\alpha}$, $u^{(2)} \in L^\infty\left(0, T; C^{1,\alpha}\right)$ (which exists) and $u^{(2)} \in L^\infty\left(0, T; L^2\right)$ (satisfying a certain energy property).

Of course the stability also follows by the same principle: if

$$u^{(n)}(0) \xrightarrow{L^2} u(0)$$

with $u(0) \in C^{1,\alpha}$ and $u \in L^\infty\left(0, T; C^{1,\alpha}\right)$ is the unique solution associated to $u(0)$, and $u^{(n)}$ are (even weaker) solutions associated to $u^{(n)}(0)$, then from

$$\left\| u^{(n)}(t) - u(t) \right\|_{L^2}^2 \leq \left\| u^{(n)}(0) - u(0) \right\|_{L^2}^2 e^{2T\|\nabla u\|_\infty}$$

we deduce the convergence

$$u^{(n)}(t) \xrightarrow{L^2} u(t) .$$

The reason why the weak-strong uniqueness principle could be relevant for convergence of particles to PDEs, in the deterministic case, is that

1. (usually, in particular for point vortices) the empirical measure of the particles, here

$$\omega_N(dx,t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}(dx)$$

   is already a very weak solution of the limit PDE
2. at time $t=0$ we assume convergence of $\omega_N(dx,0)$ to $\omega(x,0)\,dx$.

The idea is great but there is a fundamental difficulty, in the case of point vortices and 2D Euler equations.

The difficulty is that, with

$$\omega_N(dx,t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}(dx)$$

the corresponding velocity field

$$u_N(x,t) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2\pi}\frac{(x-x_i(t))^{\perp}}{|x-x_i(t)|^2}$$

is not of class L^2. Hence we cannot perform the weak-strong uniqueness computation above.
This is why Duerinckx SIMA '16, Serfaty Duke '20, Rosenzweig ARMA '22 apply the modulated energy approach, a method where an analog of the energy $\left\|u^{(n)}(t) - u(t)\right\|_{L^2}^2$, but defined on particles, is introduced.

However, let us stay a bit more on the topic of weak-strong estimates and improve the estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \leq \left\|\nabla u^{(2)}\right\|_\infty \|u\|_{L^2}^2 \qquad u = u^{(1)} - u^{(2)}$$

with the purpose to assume only

$$\left\|\omega^{(2)}\right\|_\infty < \infty.$$

Recall that $\left\|\nabla u^{(2)}\right\|_\infty < \infty$ holds in the Kato class $\omega \in C^\alpha$, while $\left\|\omega^{(2)}\right\|_\infty < \infty$ in the more general Yudovich class.

Remark: This is precisely the difference between Serfaty Duke '20, and Rosenzweig ARMA '22.

Let us apply Yudovich method:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\left\langle u \cdot \nabla u^{(2)}, u \right\rangle$$
$$\leq \left\|\nabla u^{(2)}\right\|_{L^p} \left(\int_D |u|^{2q}\, dx\right)^{1/q}$$
$$\leq \left\|\nabla u^{(2)}\right\|_{L^p} \|u\|_\infty^{\frac{2q-2}{q}} \left(\int_D |u|^2\, dx\right)^{1/q}$$

with $\frac{1}{p}+\frac{1}{q}=1$, namely

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq 2\left\|\nabla u^{(2)}\right\|_{L^p} \|u\|_\infty^{\frac{2q-2}{q}} \|u\|_{L^2}^{2\left(1-\frac{1}{p}\right)}$$

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq 2\left\|\nabla u^{(2)}\right\|_{L^p}\|u\|_{\infty}^{\frac{2q-2}{q}}\|u\|_{L^2}^{2\left(1-\frac{1}{p}\right)}$$

$$\left\|\nabla u^{(2)}\right\|_{L^p} \leq C_1 p \qquad C_1 \text{ depending only on } \|\omega_0\|_\infty$$

$$\|u\|_{\infty}^{\frac{2q-2}{q}} \leq C_2 \qquad \text{depending only on } \|\omega_0\|_\infty$$

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq Cp\|u\|_{L^2}^{2\left(1-\frac{1}{p}\right)}.$$

By comparison with the ODE

$$z' = Cpz^{1-\frac{1}{p}}$$

$$z^{\frac{1}{p}} = Ct$$

we get

$$\|u(t)\|_{L^2}^2 \leq \left(\|u_0\|_{L^2}^{2/p} + Ct\right)^p.$$

From

$$\|u(t)\|_{L^2}^2 \leq \left(\|u_0\|_{L^2}^{2/p} + Ct\right)^p$$

- if $\|u_0\|_{L^2}^{2/p}$ is small
- t is small
- p is large

we deduce $\|u(t)\|_{L^2}^2$ small.
Yudovich uniqueness theorem: $\|u_0\|_{L^2} = 0$.
Stability estimate: a bit more difficult, but possibile.

Summary

- We have seen Kato weak-strong computation, and Yudovich uniqueness computation
- Very strong PDE arguments to estimate (quantitatively) the difference of solutions
- But we cannot apply them directly to compare point vortices and limit smooth solutions
- since the velocity u of point vortices is not of class L^2.
- Serfaty Duke '20, and Rosenzweig ARMA '22 bypass this difficulty replacing $\|u(t)\|_{L^2}^2$ by a quantity based only on particles (a sort of renormalized energy, without the infinities)
- With my student Fabio Bordigoni we are performing the classical PDE arguments on the difference

$$u = u_{N,\epsilon} - u_\infty$$

between the limit solution u_∞ and the ϵ-vortex-patches approximation of point vortices.

$$W\left(\mu_t^N,\omega_t\right)\leq W\left(\mu_t^N,\omega_t^{N,\varepsilon}\right)+W\left(\omega_t^{N,\varepsilon},\omega_t\right)$$

$$\mu_t^N=\frac{1}{N}\sum_{i=1}^N\delta_{X_t^i}(dx)$$

$$\omega_t^{N,\varepsilon}=\frac{1}{N}\sum_{i=1}^N\frac{1}{\pi\epsilon^2}1_{S_\epsilon^i(t)}(x)$$

ω_t limit L^∞ solution

Smallness of $W\left(\mu_t^N,\omega_t^{N,\varepsilon}\right)$ (for $\epsilon=\epsilon_N\to0$ with suitable power law) is provided by the quantitative version of Marchioro and Pulvirenti CMP '93 (Flandoli, CPDE '18).

Smallness of $W\left(\omega_t^{N,\varepsilon},\omega_t\right)$ is a consequence of smallness of

$$\left\|u_t^{N,\varepsilon}-u_t\right\|_{L^2}^2$$

and the Yudovich method. Key lemma: smallness of $\left\|u_0^{N,\varepsilon}-u_0\right\|_{L^2}^2$.

The final results are comparable to Serfaty. We also cover Rosenzweig but with some restrictions that are under investigation. The proofs are perhaps more intuitive, from the PDE viewpoint.
Let us see more precisely the comparison with the Serfaty result.

Theorem (Serfaty Duke '20)

If ω belongs to the Kato class and

$$\int\int_{x\neq y} G(x,y)\left(\mu_0^N(dx)-\omega_0(x)\,dx\right)\left(\mu_0^N(dy)-\omega_0(y)\,dy\right)\to 0$$

then $\mu_t^N=\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}(dx)$ weakly converges to ω_t.

We have a complete proof of this result by the method outlined above, where the only difficulty is proving that

$$\left\| u_0^{N,\varepsilon}-u_0\right\|_{L^2}^2\to 0$$

as a consequence of the assumption.

The scheme is:

0 Assume $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i^0} \rightharpoonup \omega_0$

1 Choose $\epsilon_N \to 0$ such that

$$\omega_{N,\epsilon_N}(0) = \sum_{i=1}^{N} \omega_{\epsilon_N}^{i}(0) \stackrel{H^{-1}}{\to} \omega_0, \qquad \omega_{\epsilon_N}^{i}(0) = \frac{1}{N}\frac{1}{\pi\epsilon_N^2} 1_{B\left(x_i^0,\epsilon_N\right)}$$

2 get $u_{N,\epsilon_N}(t) \stackrel{L^2}{\to} u(t)$ from the weak-strong uniqueness computation

3 deduce $\omega_{N,\epsilon_N}(t) \rightharpoonup \omega(t)$

4 deduce $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)} \rightharpoonup \omega(t)$.

Steps 2, 3, 4 are really easy (Step 4 requires Marchioro-Pulvirenti theorem). The only difficulty is Step 1.

We have

$$
\begin{aligned}
&\int_D \left|u_{N,\epsilon_N}(x) - u_0(x)\right|^2 dx \\
&= \int_D \int_D G(x,y) \left(\omega_{N,\epsilon_N}(x) - \omega_0(x)\right)\left(\omega_{N,\epsilon_N}(y) - \omega_0(y)\right) dxdy \\
&= \int_D \int_D G(x,y)\, \omega_{N,\epsilon_N}(x)\, \omega_{N,\epsilon_N}(y)\, dxdy \\
&- 2 \int_D \int_D G(x,y)\, \omega_{N,\epsilon_N}(x)\, \omega_0(y)\, dxdy \\
&+ \int_D \int_D G(x,y)\, \omega_0(x)\, \omega_0(y)\, dxdy.
\end{aligned}
$$

The only difficulty is proving that

$$
\int_D \int_D G(x,y)\, \omega_{N,\epsilon_N}(x)\, \omega_{N,\epsilon_N}(y)\, dxdy \to \int_D \int_D G(x,y)\, \omega_0(x)\, \omega_0(y)\, dxdy
$$

The term

$$\int_D \int_D G(x,y)\, \omega_{N,\epsilon_N}(x)\, \omega_{N,\epsilon_N}(y)\, dxdy$$

is made of a mutual interaction term (particles $i \neq j$) which is controlled by Serfaty assumption

$$\int \int_{x \neq y} G(x,y) \left(\mu_0^N(dx) - \omega_0(x)\, dx \right) \left(\mu_0^N(dy) - \omega_0(y)\, dy \right) \to 0.$$

And a self-interaction term, the one discarded in the method of Duerinckx SIMA '16 and Serfaty Duke '20.
The self-interaction term is infinite for true point vortices.
It is finite and goes to zero for vortex patches:

$$\lim_{N\to\infty} \int_D \int_D G(x,y) \sum_{n=1}^{N} \left(\frac{1}{N\pi\epsilon_N^2} \right)^2 1_{B(X_0^n,\epsilon_N)}(x)\, 1_{B(X_0^n,\epsilon_N)}(y)\, dxdy = 0.$$

Given x,

$$\int_D |G(x,y)| 1_{B(X_0^n,\epsilon_N)}(y)\, dy \leq \int_{B(x,\epsilon_N)} |G(x,y)|\, dy$$
$$= 2\pi \int_0^{\epsilon_N} |\log r|\, rdr$$
$$\leq C\epsilon_N^2 (|\log \epsilon_N| + 1)$$

hence

$$\left| \int_D \int_D G(x,y) \sum_{n=1}^N \left(\frac{1}{N\pi\epsilon_N^2} \right)^2 1_{B(X_0^n,\epsilon_N)}(x)\, 1_{B(X_0^n,\epsilon_N)}(y)\, dxdy \right|$$
$$\leq C\epsilon_N^2 (|\log \epsilon_N| + 1) \int_D \sum_{n=1}^N \left(\frac{1}{N\pi\epsilon_N^2} \right)^2 1_{B(X_0^n,\epsilon_N)}(x)\, dx$$
$$\leq C \frac{|\log \epsilon_N| + 1}{\pi N}.$$

Concerning Rosenzweig result in the Yudovich class, we adapted the PDE proof and solved all steps except that the estimate

$$\left\| u_t^{N,\epsilon_N} - u_t \right\|_{L^2}^2 \leq \left(\left\| u_0^{N,\epsilon_N} - u_0 \right\|_{L^2}^{2/p} + C_{N,p} t \right)^p$$

gives us only local-in-time results and we have not yet discovered the trick to replicate on intervals with divergent sums.

Remark: Rosenzweig controls

$$\left\| \mu_t^N - u_t \right\|_{H^{-\delta}}^2$$

in terms of $\left\| \mu_0^N - u_0 \right\|_{H^{-\delta}}^2$. He does not use the Yudovich approach but an alternative method based on the log-Lipschitz flow. So, it is possible that the log-Lipschitz flow approach is stronger than the PDE trick of Yudovich.

Thank you!