Rotational invariance of an integration by parts formula and Lie symmetries of SDEs

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The stochastic rotational invariance of an integration by parts formula[1] inspired by the Bismut approach to Malliavin calculus is proved in the framework of the Lie symmetry theory of stochastic differential equations[2]. A new class of symmetries for an SDE is also introduced and discussed, which turns out to be the most general notion of invariance under which the integration by parts formula holds.

Joint work with F.C. De Vecchi, P. Morando, S. Ugolini.

[1] De Vecchi, F. C., Morando, P., and Ugolini, S. (2025). *Integration by parts formulas and Lie's symmetries of SDEs. Electronic Journal of Probability*, **30**.

[2] Dehò, S., De Vecchi, F. C., Morando, P., and Ugolini, S. (2025). *Random rotational invariance within a Bismut-type approach to integration by parts formulas. arXiv preprint*, arXiv:2506.12345.

Stochastic transformations

De Vecchi, F. C., Morando, P., and Ugolini, S. (2020). *Symmetries of stochastic differential equations using Girsanov transformations. Journal of Physics A: Mathematical and Theoretical.*

- Diffeomorphisms $\Phi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}_+$
- Deterministic time changes $f(t) = \int_0^t \eta(s)ds$
- Random rotations of Brownian motion $B : \mathbb{R}^n \times \mathbb{R}_+ \to SO(m)$
- Girsanov-type changes of measure via $h : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$.

Theorem: structure is preserved

Let $T = (\Phi, B, \eta, h)$ be a stochastic transformation, and (X, W) solution to $SDE_{\mu,\sigma}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, $P_T(X, W) = (P_T(X), P_T(W))$ is a solution to $E_T(SDE_{\mu,\sigma}) = SDE_{E_T(\mu),E_T(\sigma)}$ on $(\Omega, \mathcal{F}', \mathbb{Q})$, where

$$P_T(X_t) = \Phi(X_{f^{-1}(t)}); \quad P_T(W_t) = W'_{f^{-1}(t)}; \quad dW'_t = \sqrt{\eta(t)}B(X_t, t)(dW_t - h(X_t, t)dt);$$

$$E_T(\mu) = \left(\frac{1}{\eta}[L(\Phi) + D(\Phi)\sigma h]\right) \circ \Phi^{-1}; \qquad E_T(\sigma) = \left(\frac{1}{\sqrt{\eta}} D(\Phi)\sigma B^{-1}\right) \circ \Phi^{-1}.$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{F}_t} = \exp\left(\int_0^T h_\alpha(X_s, s)dW_s^\alpha - \frac{1}{2}\int_0^T (h_\alpha(X_s, s))^2 ds\right)$ and $\mathcal{F}'_t = \mathcal{F}_{f^{-1}(t)}$.

Algebraic structure of stochastic transformations

- Consider the group G given by the semidirect product $((SO(m) \times \mathbb{R}_+), \times) \ltimes_\psi (\mathbb{R}^m, +)$ with product law induced by the homomorphism $\psi : (SO(m) \times \mathbb{R}_+) \to Aut(\mathbb{R}^m)$ that maps $(B, \eta) \mapsto \psi_{(B,\eta)}$, where $\psi_{(B,\eta)} : h \mapsto \psi_{(B,\eta)}(h) := \sqrt{\eta} B^{-1} h$.
- Study the isomorphisms F between suitable trivial principal bundles $F : \mathbb{R}^n \times G \to \mathbb{R}^\ltimes \times G$
- There is a natural bijection between the set of stochastic transformations and the set of isomorphisms between principal bundles $M \times G$, since one can identify every stochastic transformation $T = (\Phi, B, \eta, h)$ with the isomorphism $F = F_T$ s.t. $F_T((x,t), g) = (\Phi(x,t), (B(x,t), \eta(t), h(x,t)) \cdot g)$.
- In order to define composition and inversion among the set of stochastic transformations and endow it with a group structure, exploit the previous correspondence and identify the isomorphisms $F_{T_1} \circ F_{T_2} \equiv F_{T_1 \circ T_2}$ and $F_T^{-1} \equiv F_{T^{-1}}$, so that

$$T_2 \circ T_1 := \left(\Phi_2 \circ \Phi_1,\ (B_2 \circ \Phi_1) B_1,\ (\eta_2 \circ f_1)\eta_1,\ \sqrt{\eta_1} B_1^{-1}(h_2 \circ \Phi_1) + h_1\right)$$
$$T_1^{-1} := (\Phi_1^{-1},\ (B_1 \circ \Phi_1^{-1})^{-1},\ (\eta_1 \circ f_1^{-1})^{-1}, -\tfrac{1}{\sqrt{\eta_1}} B_1 h_1 \circ \Phi_1^{-1})$$

Theorem: probabilistic counterpart of the algebraic description

Let T and T' be two stochastic transformations. Given a weak solution (X, W) to $SDE_{\mu,\sigma}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then in the probability space $(\Omega, \mathcal{F}', \mathbb{Q})$ it holds that

$$P_{T'} \circ P_T(X, W) = P_{T' \circ T}(X, W); \qquad E_{T'} \circ E_T(\mu, \sigma) = E_{T' \circ T}(\mu, \sigma)$$

Transition to Lie algebras

- The set of principal bundle isomorphisms $\{F_T\}_T$ forms a one-parameter group indexed by the group of stochastic transformations: $F_T \circ F_{T'} = F_{T \circ T'}, \quad F_T^{-1} = F_{T^{-1}}, \quad F_{\mathrm{id}} = \mathrm{Id}.$
- This structure transfers to $\{T_\lambda\}_\lambda$, as a one-parameter group of stochastic transformations indexed by $(\mathbb{R}, +)$: $T_{\lambda_1+\lambda_2} = T_{\lambda_1} \circ T_{\lambda_2}, \quad T_{-\lambda} = T_\lambda^{-1}, \quad T_0 = \mathrm{Id}.$
- The corresponding Lie algebra has elements $V = (Y, C, \tau, H)$ which are obtained in the usual way: $Y = \frac{d}{d\lambda}\Phi_\lambda\Big|_{\lambda=0}, \quad C = \frac{d}{d\lambda}B_\lambda\Big|_{\lambda=0}, \quad \tau = \frac{d}{d\lambda}\eta_\lambda\Big|_{\lambda=0}, \quad H = \frac{d}{d\lambda}h_\lambda\Big|_{\lambda=0}.$ We refer to such a quadruple $V = (Y, C, \tau, H)$ as an **infinitesimal stochastic transformation**.

Invariance properties and symmetries of SDEs

A symmetry is a transformation T that leaves the SDE invariant. Within this framework, different notions of invariance give rise to different notions of symmetry:

- **Strong symmetries:** The set of strong solutions is preserved

$$T \ is \ a \ strong \ transformation \ and \ E_T(SDE_{\mu,\sigma}) = SDE_{\mu,\sigma}$$

- **Weak symmetries:** The set of weak solutions is preserved

$$T \ is \ a \ weak \ transformation \ and \ E_T(SDE_{\mu,\sigma}) = SDE_{\mu,\sigma}$$

  [F.C. De Vecchi, P. Morando, S. Ugolini (2020)]

- **$\mathcal{G}$-weak symmetries:** The law of the solution process is preserved

$$E_T(L) = L$$

  [S. Dehò, F.C. De Vecchi, P. Morando, S. Ugolini (2025)]

Theorem: Finite determining equations

- $T = (\Phi, I_m, 1, 0)$ is a strong symmetry if and only if

$$\mu = \left(\frac{1}{\eta}[L(\Phi)]\right) \circ \Phi^{-1}, \qquad \sigma = \left(D(\Phi)\sigma\right) \circ \Phi^{-1}.$$

- $T = (\Phi, B, \eta, h)$ is a weak symmetry if and only if

$$\mu = \left(\frac{1}{\eta}[L(\Phi) + D(\Phi)\sigma h]\right) \circ \Phi^{-1}, \qquad \sigma = \left(\frac{1}{\sqrt{\eta}}D(\Phi)\sigma B^{-1}\right) \circ \Phi^{-1}.$$

- $T = (\Phi, B, \eta, h)$ is a $\mathcal{G}-$weak symmetry if and only if

$$\mu = \left(\frac{1}{\eta}[L(\Phi)+D(\Phi)\sigma h]\right)\circ\Phi^{-1}, \quad \sigma\sigma^T = \left(\frac{1}{\eta}D(\Phi)\sigma\sigma^T D(\Phi)^T\right)\circ\Phi^{-1}.$$

Theorem: Infinitesimal determining equations

- $V = (Y, \underline{0}_m, 0, \underline{0})$ is an infinitesimal strong symmetry if and only if

$$Y(\mu) - L(Y), \qquad [Y, \sigma] = 0.$$

- $V = (Y, C, \tau, H)$ is an infinitesimal weak symmetry if and only if

$$Y(\mu) - L(Y) - \sigma H + \tau\mu = 0, \qquad [Y, \sigma] + \frac{1}{2}\tau\sigma + \sigma C = 0.$$

- $V = (Y, C, \tau, H)$ is an infinitesimal $\mathcal{G}-$weak symmetry if and only if

$$Y(\mu) - L(Y) - \sigma H + \tau\mu = 0, \qquad [Y, \sigma\sigma^T] + \tau\sigma\sigma^T = 0.$$

Rotational invariance of the integration by parts formula

The rotation-by-parts formula derived in [3] is invariant under rotations. The inclusion of rotations among the considered transformations requires new regularity assumptions and involves nontrivial modifications to the analytical results needed for its rigorous derivation.

Hypothesis *: *For every solution* X_t *of* $SDE_{\mu,\sigma}$ *with deterministic initial conditions,*

$$\frac{1}{\eta_\lambda} C_{\alpha,\alpha} H_\alpha(X_t,t), H_\alpha(X_t,t), Y(H_\alpha)(X_t,t), L(Y^i)(X_t,t),$$
$$\Sigma_\alpha(Y^i)(X_t,t), L(Y(Y^i))(X_t,t), \Sigma_\alpha(Y(Y^i))(X_t,t) \in L^2(\mathbb{P}).$$

Theorem: Integration by Parts Formula

Let $V = (Y, C, \tau, H)$ be an infinitesimal symmetry with one-parameter group $T_\lambda = (\Phi_\lambda, B_\lambda, \eta_\lambda, h_\lambda)$ for $SDE_{\mu,\sigma}$, let $m(t) = \int_0^t \tau(s)ds$, and let F be a bounded functional with bounded first and second derivatives. Under hypothesis *, we have that

$$-m(t)\mathbb{E}_{\mathbb{P}}[L(F)(X_t)] + \mathbb{E}_{\mathbb{P}}[F(X_t)\int_0^t (H_\alpha(X_s)dW_s^\alpha)] + \mathbb{E}_{\mathbb{P}}[Y(F)(X_t)] - \mathbb{E}_{\mathbb{P}}[Y(F)(X_0)] = 0.$$

[3] De Vecchi, F. C., Morando, P., and Ugolini, S. (2025). *Integration by parts formulas and Lie's symmetries of SDEs. Electronic Journal of Probability*, **30**.

Sketch of the proof

- $\mathbb{E}_{\mathbb{P}}[F(X_t)] = \mathbb{E}_{\mathbb{Q}_\lambda}[F(P_{T_\lambda}(X_t))]$ (Invariance properties of symmetries);

- $\mathbb{E}_{\mathbb{P}}[F(X_t)] = \mathbb{E}_{\mathbb{Q}_\lambda}\left[\int_0^t L(F)(P_{T_\lambda}(X_s))ds + \partial_i(F)(X^\lambda_{s_\lambda})\sigma^i_\alpha dP_{T_\lambda}(W_s)\right]$

The addition of B only causes a variation in $P_{T_\lambda}(W_s)$, but the term containing it cancels out.

- $\mathbb{E}_{\mathbb{P}}[F(X_t)] = \mathbb{E}_{\mathbb{P}}\left[\underbrace{\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}|_{\mathcal{F}_T}}_{Z_\lambda} \int_0^{f-\lambda(t)} L(F)\circ\Phi_\lambda(X_s,s)f'_\lambda(s)ds\right]$ $\quad (\circ)$.

$\partial_\lambda Z_\lambda$ would change with the addition of B, but it is evaluated at $\lambda = 0$ and $B_0 = I_m$.

by time change, definition of $P_{T_\lambda}(X_t)$ and Radon-Nikodym theorem. Deriving $(\circ)$ with respect to λ and evaluating the result in $\lambda = 0$ we get

$$0 = \mathbb{E}_{\mathbb{P}}\left[\int_0^T H_\alpha(X_t)dW_t^\alpha \int_0^t L(F)(X_s)ds\right] - m(t)\mathbb{E}_{\mathbb{P}}[L(F)(X_t)] + \mathbb{E}_{\mathbb{P}}\left[\int_0^t Y(L(F))(X_s)ds\right] + \mathbb{E}_{\mathbb{P}}\left[\int_0^t \tau(s)L(F)(X_s)ds\right]$$

$\mathcal{G}$-weak symmetries and invariance under rotation

- Since Brownian motion is rotational-invariant, given a smooth function $B : \mathbb{R}^n \times \mathbb{R}_+ \to SO(m)$ and a weak solution (X, W) to $SDE_{\mu,\sigma}$, then (X, W') is a weak solution to $SDE_{\mu,\sigma B^{-1}}$, where $W'_t = \int_0^t B(X_s, s) dW_s$.
- The inclusion of the Brownian motion rotations among the class of admissible random transformations of an SDE implies that the law of a solution to $SDE_{\mu,\sigma}$ is no longer identified by coefficients (μ, σ) but by the whole family

$$\mathcal{G}auge(SDE_{\mu,\sigma}) := (SDE_{\mu,\sigma B^{-1}})_{B \in SO(m)}$$

- The novel notion of $\mathcal{G}$-weak symmetry encodes this invariance property and shifts the focus from the weak solution (X, W) to the diffusion process X associated with the martingale problem $(\mu, \sigma\sigma^T)$. Moreover, it represents the most general symmetry concept under which the integration by parts formula remains valid.

Example: 2-dimensional Brownian motion

Consider a 2-dimensional Brownian motion, obeying the SDE

$$\begin{pmatrix} dX_t \\ dY_t \\ dZ_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mu} dt + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\sigma} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}$$

where the trivial extra component Z represents the evolution of time. Solving the infinitesimal determining equations, we find two infinite-dimensional families of symmetries V_α and V_β:

$$V_\alpha = \left(\begin{pmatrix} \frac{1}{2}\alpha(z)x \\ \frac{1}{2}\alpha(z)y \\ \int \alpha(z)dz \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \alpha(z), \begin{pmatrix} -\frac{1}{2}x\alpha'(z) \\ -\frac{1}{2}y\alpha'(z) \end{pmatrix} \right),$$

$$V_\beta = \left(\begin{pmatrix} \beta(z)y \\ -\beta(z)x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta(z) \\ -\beta(z) & 0 \end{pmatrix}, 0, \begin{pmatrix} -y\beta'(z) \\ x\beta'(z) \end{pmatrix} \right).$$

related respectively to the random time and the random rotation invariance of Brownian motion and depending on arbitrary functions of time α and β.

- If $B(t) \in SO(m)$, then $W'_t = \int_0^t B(t) dW_t$ is still a $\mathbb{P}-$Brownian motion.
  The symmetry V_β provides a generalization of the previous classical result, considering the process $\tilde{W}_t = B(t) W_t$. Since W'_t is already a $\mathbb{P}$-Brownian motion, $\tilde{W}$ is still a Brownian motion with respect to a new probability measure $\mathbb{Q}$, whose Radon-Nikodym density is given by
  $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\int_0^t B'(s) W_s \cdot dW_s - \frac{1}{2} \int_0^T |B'(s) \cdot W_s|^2 ds \right)$. Indeed,

$$d\tilde{W}_t = d(B(t)W_t) = B(t)dW_t + B'(t)W_t dt = B(t)dW_t - (-B'(t)W_t)dt = dW'_t - \underbrace{(-B'(t)W_t)}_{h} dt.$$

- Symmetry V_β encodes this invariance property. Indeed, from classical Lie theory tools (flow reconstruction), V_β is related to a finite transformation $T_\lambda = (\Phi_\lambda, B_\lambda, \eta_\lambda, h_\lambda)$ where

$$\Phi_\lambda = \begin{pmatrix} \cos(\beta(z)\lambda) \cdot x + \sin(\beta(z)\lambda) \cdot y \\ -\sin(\beta(z)\lambda) \cdot x + \cos(\beta(z)\lambda) \cdot y \\ z \end{pmatrix}; B_\lambda = \begin{pmatrix} \cos(\beta(z)\lambda) & \sin(\beta(z)\lambda) \\ -\sin(\beta(z)\lambda) & \cos(\beta(z)\lambda) \end{pmatrix}; \eta_\lambda = 1; h_\lambda = \begin{pmatrix} -\beta'(z)\lambda \cdot y \\ \beta'(z)\lambda \cdot x \end{pmatrix}.$$

  Since V_β is a symmetry, by construction we have that $(P_{T_\lambda}(X, W))$ solves the same SDE as (X, W), that is

$$\underbrace{d\begin{pmatrix} \cos(\beta\lambda) \cdot X_t + \sin(\beta\lambda) \cdot Y_t \\ -\sin(\beta\lambda) \cdot X_t + \cos(\beta\lambda) \cdot Y_t \\ Z_t \end{pmatrix}}_{dP_{T_\lambda}(X) = d\tilde{W}_t} = \underbrace{\begin{pmatrix} \cos(\beta\lambda) & \sin(\beta\lambda) \\ -\sin(\beta\lambda) & \cos(\beta\lambda) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} - \begin{pmatrix} \beta'\lambda \sin(\beta\lambda) \cdot X_t - \beta'\lambda\cos(\beta\lambda) \cdot Y_t \\ \beta'\lambda\cos(\beta\lambda) \cdot X_t + \beta'\lambda \sin(\beta\lambda) \cdot Y_t \\ 1 \end{pmatrix} dt}_{dP_{T_\lambda}(W) = dW'_t - h\,dt}$$

  In particular, (X, W) under $\mathbb{P}$ has the same law as $(P_{T_\lambda}(X, W))$ under $\mathbb{Q}_\lambda$: $P_{T_\lambda}(X) = \tilde{W}$ is a $\mathbb{Q}_\lambda-$Brownian motion.

The corresponding integration by parts formulas, valid for any arbitrary functions of time α and β are:

$$\int_0^t \alpha(z)dz \cdot \mathbb{E}_{\mathbb{P}}\Big[\frac{1}{2}D^2F(X_t,Y_t)\Big] = \mathbb{E}_{\mathbb{P}}\Big[F(X_t,Y_t)\int_0^t -\frac{1}{2}X_t\alpha'(z)dW_t^1 + \frac{1}{2}Y_t\alpha'(z)dW_t^2\Big] +$$

$$+\mathbb{E}_{\mathbb{P}}\Big[\frac{1}{2}\alpha(t)X_t\partial_x F(X_t,Y_t) + \frac{1}{2}\alpha(t)Y_t\partial_y F(X_t,Y_t)\Big],$$

$$0 = \mathbb{E}_{\mathbb{P}}\Big[F(X_t,Y_t)\int_0^t -Y_s\beta'(z)dW_s^1 + X_s\beta'(z)dW_s^2\Big] + \mathbb{E}_{\mathbb{P}}\Big[\beta(t)Y_t\partial_x F(X_t,Y_t) - \beta(t)X_t\partial_y F(X_t,Y_t)\Big].$$

In particular, by choosing α and β to be nonzero **nonzero constants**, we obtain respectively:

$$t \cdot \mathbb{E}_{\mathbb{P}}\Big[\partial_x^2 F(X_t,Y_t) + \partial_y^2 F(X_t,Y_t)\Big] = \mathbb{E}_{\mathbb{P}}\Big[X_t\partial_x F(X_t,Y_t) + Y_t\partial_y F(X_t,Y_t)\Big],$$

$$\mathbb{E}_{\mathbb{P}}\Big[Y_t\partial_x F(X_t,Y_t)\Big] = \mathbb{E}_{\mathbb{P}}\Big[X_t\partial_y F(X_t,Y_t)\Big],$$

which correspond precisely to **Stein's lemma** and Wick's theorem, applied to the solution (X_t, Y_t) of the SDE above.

Example: a stochastic Lotka Volterra model

Consider now the following SDE:

$$\begin{pmatrix} dX_t \\ dY_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} X_t(\alpha - \beta Y_t) \\ Y_t(\delta X_t - \gamma) \\ 1 \end{pmatrix} dt + \begin{pmatrix} \tilde{\sigma} X_t & 0 \\ 0 & \tilde{\sigma} Y_t \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \quad (1)$$

which can be seen as the well known predator-prey Lotka Volterra model plus a stochastic noise (see Arató, M. (2003). *Mathematical and Computer Modelling*).
Solving the determining equations, we find the following two-infinite dimensional family of symmetries

$$V_a = \left(\begin{pmatrix} \frac{1}{2} a(z) x \ln x \\ \frac{1}{2} a(z) y \ln y \\ \int a(z) dz \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a(z), \begin{pmatrix} -\frac{1}{4} a(z)\tilde{\sigma} + \frac{1}{2\tilde{\sigma}} a(z)(\alpha - \beta y) - \frac{1}{2\tilde{\sigma}} a'(z) \ln x - \frac{1}{2\tilde{\sigma}} \beta a(z) y \ln y \\ -\frac{1}{4} a(z)\tilde{\sigma} + \frac{1}{2\tilde{\sigma}} a(z)(\delta x - \gamma) - \frac{1}{2\tilde{\sigma}} a'(z) \ln y + \frac{1}{2\tilde{\sigma}} \delta a(z) x \ln x \end{pmatrix} \right);$$

$$V_b = \left(\begin{pmatrix} b(z) x \ln y \\ -b(z) y \ln x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & b(z) \\ -b(z) & 0 \end{pmatrix}, 0, \begin{pmatrix} -\frac{1}{2}\tilde{\sigma} b(z) + \frac{b(z)}{\tilde{\sigma}}(\delta x - \gamma) - \frac{1}{\tilde{\sigma}} b'(z) \ln y + \frac{\beta}{\tilde{\sigma}} b(z) y \ln x \\ \frac{1}{2}\tilde{\sigma} b(z) - \frac{b(z)}{\tilde{\sigma}}(\alpha - \beta y) + \frac{1}{\tilde{\sigma}} b'(z) \ln x + \frac{\delta}{\tilde{\sigma}} b(z) x \ln y \end{pmatrix} \right).$$

We obtain the following integration by parts formulas for V_a

$$-\int a(z)dz\mathbb{E}_{\mathbb{P}}[X_t(\alpha-\beta Y_t)F_x(X_t,Y_t)+Y_t(\delta X_t-\gamma)F_y(X_t,Y_t)+\frac{\tilde{\sigma}^2X_t^2}{2}F_{xx}(X_t,Y_t)+\frac{\tilde{\sigma}^2Y_t^2}{2}F_{yy}]+$$

$$+\mathbb{E}_{\mathbb{P}}\Big[F(X_t,Y_t)\int_0^t\Big(-\frac{a(s)\tilde{\sigma}}{4}+\frac{a(s)}{2\tilde{\sigma}}(\alpha-\beta Y_s)-\frac{1}{2\tilde{\sigma}}a'(s)\ln(X_s)-\frac{1}{2\tilde{\sigma}}\beta a(s)Y_s\ln(Y_s)\Big)dW_s^1+$$

$$\Big(-\frac{a(s)\tilde{\sigma}}{4}+\frac{a(s)}{2\tilde{\sigma}}(\delta X_s-\gamma)-\frac{1}{2\tilde{\sigma}}a'(s)\ln(Y_s)+\frac{1}{2\tilde{\sigma}}\delta a(s)X_s\ln(X_s)\Big)\Big]+$$

$$+\mathbb{E}_{\mathbb{P}}[\frac{a(t)}{2}X_t\ln(X_t)F_x(X_t,Y_t)+\frac{a(t)}{2}Y_t\ln(Y_t)F_y(X_t,Y_t)]+$$

$$-\mathbb{E}_{\mathbb{P}}[\frac{a(t)}{2}X_t\ln(X_t)F_x(X_0,Y_0)+\frac{a(t)}{2}Y_t\ln(Y_t)F_y(X_0,Y_0)]=0$$

and for V_b

$$\mathbb{E}_{\mathbb{P}}\Big[F(X_t,Y_t)\int_0^t\Big(-\frac{\tilde{\sigma}b(s)}{2}+\frac{b(s)}{\tilde{\sigma}}(\delta X_s-\gamma)-\frac{b'(s)}{\tilde{\sigma}}\ln(Y_s)+\frac{\beta b(s)}{\tilde{\sigma}}Y_s\ln(X_s)\Big)dW_s^1+$$

$$+\Big(\frac{\tilde{\sigma}b(s)}{2}-\frac{b(s)}{\tilde{\sigma}}(\alpha-\beta Y_s)+\frac{b'(s)}{\tilde{\sigma}}\ln(X_s)+\frac{\delta b(s)}{\tilde{\sigma}}X_s\ln(Y_s)\Big)dW_s^2\Big]+$$

$$+\mathbb{E}_{\mathbb{P}}[b(t)X_t\ln(Y_t)F_x(X_t,Y_t)-b(t)Y_t\ln(X_t)F_y(X_t,Y_t)]+$$

$$-\mathbb{E}_{\mathbb{P}}[b(t)X_t\ln(Y_t)F_x(X_0,Y_0)-b(t)Y_t\ln(X_t)F_y(X_0,Y_0)]=0.$$

Main references I

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Thanks for your attention!

Preliminary analytical results

- Under suitable hypoteses the family of stochastic integrals $(\int_0^t g_\lambda(r)dW_r)_t$ admits a modification differentiable with continuity w.r.t. λ a.e. s.t. $\partial_\lambda^i \int_0^t g_\lambda(r)dW_r = \int_o^t \partial_\lambda^i g_\lambda(r)dW_r \;\; \forall(t,\lambda) \; a.e.$
- If there exists a number $M = M_t > 0$ and a non-negative function $\phi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ such that ϕ is a Lyapunov function on $[0,T] \times \mathbb{R}^n$, then the martingale problem associated with the infinitesimal generator L is well-posed, the solution process is non-explosive and $\mathbb{E}_x[\phi(t,X_t)] \leq exp(Mt)\phi(0,X_0), \quad t \geq 0$.
- Time change formula for Itô integrals: If $\eta(s,\omega)$, $B(s,\omega)$ and $\alpha(s,\omega)$ are s-continuous, $\alpha(0,\omega) = 0$ for a.a. ω, $\mathbb{E}[\alpha_t] < \infty$, $B(s,\omega) \in SO(m)$ and $v(s,\omega)$ bounded and s-continuous, then $\int_0^{f_{-\lambda}(t)} v(s,\omega)dW_s = \int_0^t v(f_{-\lambda}(s),\omega)\frac{1}{\sqrt{\eta(f_{-\lambda}(s))}}B^{-1}(f_{-\lambda}(s),\omega)f_{-\lambda}(s)d\tilde{W}_s \;\; \mathbb{P}-a.s.$ where $\tilde{W}_t = \lim_{j\to\infty}\sum_j \sqrt{\eta(f_{-\lambda}(j))}B(f_{-\lambda}(j))\Delta W_{f_{-\lambda}(j)} = \int_0^{f_{-\lambda}(t)} \sqrt{\eta(s)}B(s,\omega)dW_s$.
- If $g(\lambda,X) : \mathbb{R} \times M \to \mathbb{R}$ is integrable and differentiable with continuity w.r.t. λ and s.t. $\mathbb{E}_\mathbb{P}[|\partial_\lambda^2 g(\lambda,X)|] < \infty$, then $\partial_\lambda \mathbb{E}_\mathbb{P}[g(\lambda,X)] = \mathbb{E}_\mathbb{P}[\partial_\lambda g(\lambda,X)]$.
- If $V = (Y,C,\tau,H)$ is an infinitesimal symmetry of $SDE_{\mu,\sigma}$ which verify **hypotesis***, with one-parameter group associated $T_\lambda = (\Phi_\lambda, \eta_\lambda, h_\lambda)$, then $\mathbb{E}_\mathbb{P}[|\partial_\lambda^2 P(\lambda)|] < \infty$, where $P(\lambda) = \frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}\Big(\int_0^{f_{-\lambda}(t)} L(F) \circ \Phi_\lambda(X_s,s)f'_\lambda(s)ds\Big)$.