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Stochastic optimal control for quadratic hedging in presence of memory

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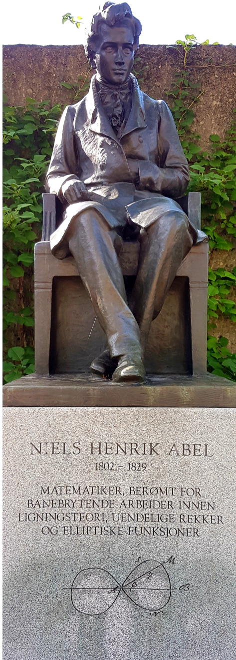
Based on works with **Anton Yurchenko-Tytarenko** and **Yuliya Mishura**



SV some features

Volatility in the Black-Scholes model is not constant. Indeed,

- realised volatility shows clustering
- negative correlation between variance and returns
- empirically observed implied volatility $(\tau, \kappa) \longrightarrow \hat{\sigma}_{emp}(\tau, \kappa)$ has a "smiley" shape, particularly when $\kappa \approx 0$ (at the money)

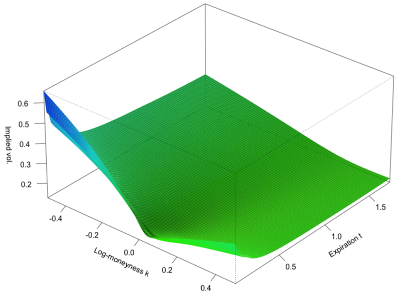


Figure: Shape of the S&P volatility surface as of June 20, 2013 volatility surface.The plot is taken from Gatheral, Jaisson, Rosenbaum (2014) [Figure 1.1]. Note the gradual levelling of volatility smile as expiration time increases.

SV and implied volatility surface

The model-generated Black-Scholes implied volatility surface

$$(\tau, \kappa) \longrightarrow \hat{\sigma}(\tau, \kappa)$$

is a benchmark for evaluating price models with the empirically observed ones $\hat{\sigma}_{emp}(\tau, \kappa)$.

Notation. Here τ is the time to maturity and $\kappa := log \frac{K}{e^{r\tau} S_0}$ is the log-moneyness with K denoting the strike, S_0 the current price of an underlying asset, and r the instantaneous interest rate.

The smile at-the-money, when $\kappa \approx 0$, becomes progressively steeper as $\tau \to 0$ with a rule-of-thumb behaviour

$$\left| \frac{\widehat{\sigma}_{\text{emp}}(\tau, \kappa) - \widehat{\sigma}_{\text{emp}}(\tau, \kappa')}{\kappa - \kappa'} \right| \propto \tau^{-\frac{1}{2}+H}, \quad \kappa, \kappa' \approx 0, \quad H \in \left(0, \frac{1}{2}\right).$$

This is known as the power law of the at-the-money implied volatility skew.

A SV model should satisfy $\left| \frac{\partial \widehat{\sigma}}{\partial \kappa}(\tau, \kappa) \right|_{\kappa=0} = O(\tau^{-\frac{1}{2}+H}), \tau \to 0.$

RK. Recall that $\hat{\sigma}(\tau, \kappa)$ is given by the solution of the equation between the theoretical Black-Scholes prices of a vanilla call option with maturity T and strike K with the observed prices on the market. This provides a volatility estimator as a function of T, K, S_0, then re-parametrised in terms of τ, κ, S_0.

Smiles and roughness ...

It is not easy to reproduce this effect.

For example, classical Brownian diffusion stochastic volatility models fail in this. See, e.g. Alòs, León and Vives (2007) or Lee (2006),

This effect can be replicated by an SV with very low Hölder regularity within the rough volatility framework popularized by Gatheral, Jaisson and Rosenbaum (2014). The efficiency of this approach is supported by:

- A theoretical result of Fukasawa (2021) suggests that the volatility process cannot be high-order Hölder continuous in a continuous non-arbitrage model exhibiting the power law property. In other words, the volatility roughness is, in some sense, a necessary condition (at least in the fully continuous setting).
- In Alòs, León and Vives, the short-term explosion of the implied volatility skew can be deduced from the explosion of the Malliavin derivative of volatility. This is a characteristic exhibited by *fractional Brownian motion with* $H < 1/2$.

... and long memory

However, for large τ, there is evidence of long memory.

In the context of fractional Brownian motion, empirical studies, as Ding et al. (1993, 1996), Bollerslev, Ole Mikkelsen (1993), Breidt et al. (1998), Cont et al. (1997, 2005, 2006), report that the autocorrelation of absolute log-returns

$$R(\tau) := \text{corr}\left(\left|\log\left(\frac{S(t+\Delta)}{S(t)}\right)\right|, \left|\log\left(\frac{S(t+\tau+\Delta)}{S(t+\tau)}\right)\right|\right),$$

is positive and decays as $O(\tau^{H-\frac{1}{2}})$, $\tau \to \infty$, with $H \geq 1/2$.

See also, Willinger, Taqqu, Teverovsky (1999), Lobato et al. (2000) - who analyse volatility in connection to trading volumes - Comte and Renault (1998), Funahashi and Kijima (2017)

RK: If we restrict to the context of one parameter fractional Brownian noise, then it seems that there is a "fractional puzzle". However, long memory and roughness are not self-excluding properties[1] .

[1] Funahashi and Kijima (2017), Alòs, García, Lorite, Gatarek (2021), Ayache, Peng (2012), Corlay, Lebovits, Lévy Véhel (2014)

Other issues and volatility bounds

- A desirable property is the positivity of its paths. Indeed, densities of martingale measures normally contain expressions of the form $\int_0^T \frac{1}{\sigma(s)} dW(s)$ and $\int_0^T \frac{1}{\sigma^2(s)} ds$. See Biagini, Guasoni, Pratelli (2000). Hence, if the volatility hits zero, there is no transparent procedure of switching between physical and pricing measures.
- A common issue is moment explosions, i.e. $\mathbb{E}[S^r(t)]$ may be infinite for all time points t after some t_*. See Andersen (2006) [Section 8]: "*Several actively traded fixed-income derivatives*[2] *require at least* L^2 *solutions to avoid infinite model prices*".

As possible solution one can take an SV model that is bounded and bounded away from zero.

Remark. It turns out that these two assumptions are fairly silently widespread in the literature. Among the multiple examples, we mention Karatzas and Ocone (1991), Fouque, Papanicolaou, Sircar, Sølna (2003), Alòs and Léon (2017), Garnier and Sølna (2020), Alòs, Rolloos, Shiraya (2022) and Rosenbaum and Zhang (2022) – where also the bounds have been used as an additional calibration parameter, called *minimal instantaneous variance*.

[2] Such as CMS swaps or Eurodollar futures.

Agenda

1. A glance into SVV models
2. Malliavin differentiability of first and of higher order
3. Implied volatility surface and power law
4. Martingale measures
5. Quadratic hedging via Markovian approximations to the volatility model.

1. A glance into SVV models

We go further suggesting Sandwiched Volterra Volatility (SVV) models, i.e.

$$S_i(t) = S_i(0) + \int_0^t \mu_i(s)S_i(s)ds + \int_0^t Y_i(s)S_i(s)dB_i^S(s),$$

$$Y_i(t) = Y_i(0) + \int_0^t b_i(s, Y_i(s))ds + \int_0^t \mathcal{K}_i(t,s)dB_i^Y(s),$$

$i = 1, ..., d$, such that

$$0 < \varphi_i(t) < Y_i(t) < \psi_i(t),$$

for some deterministic bounds.

RK: Also Merino, Pospìsil, Sobotka, Sottinen, and Vives (2021) and Abi Jaber, Illand, and Li (2022), consider an SV model driven by Gaussian Volterra processes. SVV has the sandwich feature.

SVV dynamics

$$Y_i(t) = Y_i(0) + \int_0^t b_i(s, Y_i(s))ds + \int_0^t \mathcal{K}_i(t,s)dB_i^Y(s)$$

Proposition.[3] The Z_i have Hölder continuous modification up to order $H_i \in (0,1)$ iff

$$\int_0^t (\mathcal{K}_i(t,u) - \mathcal{K}_i(s,u))^2 du \le C_\lambda |t-s|^{2\lambda}, \quad 0 \le s \le t \le T,$$

holds for any $\lambda \in (0, H_i)$. Moreover, the random variable

$$\Lambda_{\lambda,i} := \sup_{0 \le s < t \le T} \frac{|Z_i(t) - Z_i(s)|}{|t-s|^\lambda}$$

has moments of all orders.

Example: Multifractional Brownian motion. Let h: $[0, T] \to (0, 1)$ be β-Hölder continuous function. Then

$$Z(t) := \int_0^t (t-s)^{h(t)-\frac{1}{2}} dB(s)$$

is Hölder continuous of order up to $\min\left\{\frac{1}{2}, \beta, \min_{t\in[0,T]} h(t)\right\}$. See Peltier, Lévy Vehel (1995) and Azmoodeih et al. (2014).

[3] See Azmoodeh, Sottinen, Viitasaari, and Yazigi (2014)

The sandwich bounds are the couples $\varphi_i, \psi_i \colon [0, T] \to \mathbb{R}$,

- φ_i, ψ_i are Hölder continuous up to the order H_i
- $0 < \varphi_i(t) < \psi_i(t)$

Assumption

(i) $b_i \in C(\mathcal{D}^i_{0,0})$ [4]

(ii) there exist $c > 0$, $p > 1$, some y_* (this depend on the bounds' strip) such that for $\varepsilon > 0$

$$|b_i(t_1, y_1) - b_i(t_2, y_2)| \le \frac{c}{\varepsilon^p} \left(|y_1 - y_2| + |t_1 - t_2|^\lambda \right), \ (t_1, y_1), (t_2, y_2) \in \mathcal{D}^i_{\varepsilon,\varepsilon}$$

(iii) for some constant $\gamma_i > \frac{1}{H_i} - 1$, where H_i is from (Aii),

$$b_i(t, y) \ge \frac{c}{(y - \varphi_i(t))^{\gamma_i}}, \quad (t, y) \in \mathcal{D}^i_{0,0} \setminus \mathcal{D}^i_{y_*,0},$$

$$b_i(t, y) \le -\frac{c}{(\psi_i(t) - y)^{\gamma_i}}, \quad (t, y) \in \mathcal{D}^i_{0,0} \setminus \mathcal{D}^i_{0,y_*};$$

(iv) there exists continuous $\frac{\partial b_i}{\partial y}$ wrt the spatial variable and $\frac{\partial b_i}{\partial y}(t, y) < c$

[4] $\mathcal{D}^i_{a_1,a_2} := \{(t, y) \in [0, T] \times \mathbb{R}_+, \ y \in (\varphi_i(t) + a_1, \psi_i(t) - a_2)\}$, for $a_1, a_2 > 0$

Results on the SV equation.[5]

- For each $\lambda \in \left(\frac{1}{\gamma_i+1}, H_i\right)$, where γ_i is from (iii), one finds $L_{1,i}$, $L_{2,i} > 0$ and $\alpha_i > 0$, depending on b_i and λ, such that for all $t \in [0, T]$:

$$\varphi(t) + \frac{L_{1,i}}{(L_{2,i} + \Lambda_{\lambda,i})^{\alpha_i}} \le Y_i(t) \le \psi(t) - \frac{L_i^1}{(L^{2,i} + \Lambda_{\lambda,i})^{\alpha_i}},$$

  where $\Lambda_{\lambda,i}$ is given in (8). Hence the sandwich $\varphi_i(t) < Y_i(t) < \psi_i(t)$ a.s., $t \in [0, T]$.

- Since $\Lambda_{\lambda,i}$ has moments of all orders, for any $r > 0$ we have

$$\mathcal{E}\left[\sup_{t\in[0,T]} \frac{1}{(Y_i(t) - \varphi_i(t))^r}\right] < \infty, \quad \mathcal{E}\left[\sup_{t\in[0,T]} \frac{1}{(\psi_i(t) - Y_i(t))^r}\right] < \infty.$$

[5] See Di Nunno, Mishura, Yurchenko-Tytarenko (2022)

Example. Consider the sandwiched SDE of the form

$$Y(t) = Y(0) + \int_0^t \left(\frac{\kappa_1}{(Y(s) - \varphi(s))^4} - \frac{\kappa_2}{(\psi(s) - Y(s))^4} \right) ds + Z(t).$$

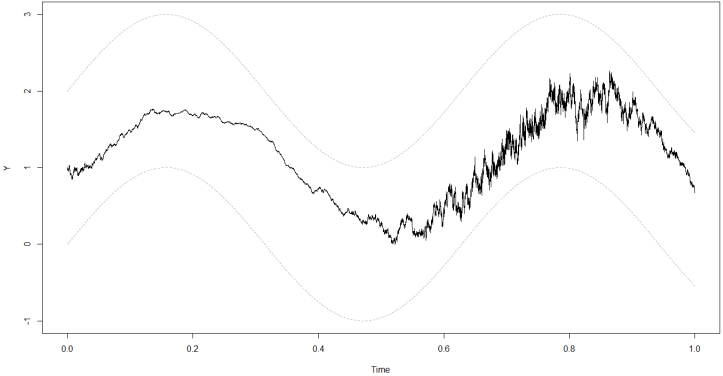


Figure: A sample path of (11) generated using the backward Euler approximation scheme; $N = 10000$, $T = 1$, $Y(0) = 1$, $\kappa_1 = \kappa_2 = 1$, $\varphi(t) = \sin(10t)$, $\psi(t) = \sin(10t) + 2$, Z is a multifractional Brownian motion with functional Hurst parameter $H(t) = \frac{1}{5}\sin(2\pi t) + \frac{1}{2}$. Method: Drift implicit Euler scheme by Di Nunno, Mishura, Yurchenko-Tytarenko (2022).

The price process

$$S_i(t) = S_i(0) + \int_0^t \mu_i(s) S_i(s) ds + \int_0^t Y_i(s) S_i(s) dB_i^S(s),$$

- $S_i(0) > 0$,
- μ_i: $[0, T] \to \mathbb{R}$, are H_i-Hölder continuous,
- Y_i, $i = 1, ..., d$ are SVV models

Theorem. For any $i = 1, ..., d$, equation (7) has a unique solution of the form

$$S_i(t) = S_i(0) \exp \left\{ \int_0^t \left(\mu_i(s) - \frac{Y_i^2(s)}{2} \right) ds + \int_0^t Y_i(s) dB_i^S(s) \right\}.$$

Furthermore, for any $r \in \mathbb{R}$: $\mathbb{E}\left[\sup_{t\in[0,T]} S_i^r(t)\right] < \infty$.

Proof. For the existence, we refer to Cohen and Elliott (2015) and the Itô formula. For the moments we exploit the boundedness of Y_i and the Novikov condition to check the uniform integrability of some exponential martingale providing then bounds to the moments.

Numéraire and discounted price process

As a numéraire, we will use

$$e^{\int_0^t \nu(s)ds}, \qquad t \in [0, T],$$

where $\nu\colon [0, T] \to \mathbb{R}_+$ denotes a Hölder continuous function of order $\min_{i=1,\dots,d} H_i$ representing an instantaneous interest rate.

The discounted price process are then

$$\widetilde{S}_i(t) := e^{-\int_0^t \nu(s)ds} S_i(t), \quad t \in [0, T], \quad i = 1, \dots, d,$$

has thus dynamics of the form

$$\widetilde{S}_i(t) = S_i(0) + \int_0^t \widetilde{\mu}_i(s)\widetilde{S}_i(s)ds + \int_0^t Y_i(s)\widetilde{S}_i(s)dB_i^S(s), \quad t \in [0, T],$$

where $\widetilde{\mu}_i := \mu_i - \nu$.

2. Malliavin differentiability in SVV

Here we proceed as follow:

1. Study the differentiability of Y_i
2. Provide a new chain rule and deduce the differentiability of S_i.
3. Study of higher order Malliavin differentiability of Y

We use Malliavin derivatives for:

a. studying of the power law
b. numerical pricing of options (focus on discontinuous payoffs using the integration-by-parts method)

We adopt the notation $\mathcal{H}$ for the Cameron-Martin space, i.e. the space of all continuous functions $F = (F_1, \dots, F_{2d}) \in C_0([0, T]; \mathbb{R}^{2d})$ such that

$$F_i(\cdot) = \int_0^{\cdot} f_i(s)ds,$$

where $f = (f_1, \dots, f_{2d}) \in L^2([0, T]; \mathbb{R}^{2d})$.

Malliavin derivative and $\mathbb{D}^{k,p}$

Let $C_p^{(\infty)}(\mathbb{R}^n)$ be the space of infinitely differentiable functions with the derivatives of at most polynomial growth. Let W be a standard Brownian motion. Define

$$W(h) := \int_0^T h(t)dW(t), \qquad h \in L^2([0,T])$$

Definition. The random variables X of the form

$$X = f(W(h_1), \dots, W(h_n)),$$

where $n \geq 1$, $f \in C_p^{(\infty)}(\mathbb{R}^n)$ and $h_1, \dots, h_n \in L^2([0,T])$ are called *smooth*. Definition. For any smooth X. The Malliavin derivative of X (with respect to W) is the $L^2([0,T])$-valued random variable

$$DX := \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(W(h_1), \dots, W(h_n))h_k.$$

The operator $D : L^p(\Omega) \Longrightarrow L^p(\Omega \times [0,T])$ is closable for any $p \geq 1$. The closure D is taken with respect to the norm

$$\|X\|_{1,p} := \left(\mathbb{E}\left[|X|^p\right] + \mathbb{E}\left[\left(\int_0^T (D_s X)^2 ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}.$$

The domain is denoted $\mathbb{D}^{1,p}$.

This definition can be iterated to introduce the iterated derivative $D^k X$ as a random variable with values in $(L^2([0,T]))^{\otimes k} \sim L^2([0,T]^k)$. One can also define $\mathbb{D}^{k,p}$ with respect to the seminorm

$$\|X\|_{k,p} := \left(\mathbb{E}\left[|X|^p\right] + \sum_{j=1}^{k} \mathbb{E}\left[\|D^j X\|^p_{L^2([0,T]^k)} \right] \right)^{\frac{1}{p}}.$$

Lemma. Let $p > 1$ and $X \in \mathbb{D}^{1,2}$ be such that

$$\mathbb{E}\left[|X|^p\right] < \infty \quad \text{and} \quad \mathbb{E}\Big[\Big(\int_0^T (D_s X)^2 ds\Big)^{\frac{p}{2}}\Big] < \infty.$$

Then $X \in \mathbb{D}^{1,p}$.

Theorem: Generalized Malliavin product rule.
Let $X_1, X_2 \in \mathbb{D}^{1,2}$ be such that

(i) $X_1 X_2 \in L^2(\Omega)$;

(ii) $X_2 D X_1, X_1 D X_2 \in L^2(\Omega \times [0, T])$.

Then $X_1 X_2 \in \mathbb{D}^{1,2}$ and $D[X_1 X_2] = X_2 D X_1 + X_1 D X_2$.

If, in addition,

$$\mathbb{E}[|X_1 X_2|^p] < \infty, \quad \mathbb{E}\Big[\Big(\int_0^T (X_2 D_u X_1 + X_1 D_u X_2)^2 du\Big)^{\frac{p}{2}}\Big] < \infty$$

for some $p \geq 2$, then $X_1 X_2 \in \mathbb{D}^{1,p}$.

The proof proceeds with a constructive approach via mollification and approximations.

The Malliavin differentiability of Y_i is studied by the criterion of Sugita (1985).

Theorem. A random variable $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ belongs to $\mathbb{D}^{1,2}$ wrt W if and only if both conditions are satisfied:

(i) η is *ray absolutely continuous*, i.e. for any $F \in \mathcal{H}$ there exists a version of the process $\{\eta(\omega + \varepsilon F),\ \varepsilon \geq 0\}$ that is absolutely continuous;

(ii) η is *stochastically Gateaux differentiable*, i.e. there exists a random vector

$$D\eta \in L^2\left(\Omega; L^2([0, T]; \mathbb{R}^{2d})\right)$$

such that for any $F = \int_0^\cdot f(s)ds \in \mathcal{H}$, $f \in L^2([0, T]; \mathbb{R}^{2d})$,

$$\frac{1}{\varepsilon}\left(\eta\left(\omega + \varepsilon F\right) - \eta(\omega)\right) \xrightarrow{\mathbb{P}} \langle D\eta, f\rangle_{L^2\left([0,T];\mathbb{R}^{2d}\right)}, \quad \varepsilon \to 0.$$

In this case $D\eta$ from (ii) is the Malliavin derivative of η.

Theorem. In this model, for all i and t, $\quad Y_i(t) \in \mathbb{D}^{1,2}$ and

$$DY_i(t) = (D^1 Y_i(t), ..., D^{2d} Y_i(t)) \in L^2\left(\Omega; L^2([0, T]; \mathbb{R}^{2d})\right),$$

$$D^j_u Y_i(t) := \ell_{d+i,j}\left(\mathcal{K}_i(t, u) + \int_u^t \mathcal{K}_i(s, u)\frac{\partial b_i}{\partial y}(s, Y_i(s))e^{\int_s^t \frac{\partial b_i}{\partial y}(v, Y_i(v))dv}ds\right)\mathbf{1}_{[0,t]}(u),$$

where $\ell_{d+i,j}$ are elements of $\mathcal{L}$.

The Malliavin differentiability of the prices S_i is studied by the representation

$$S_i(t) = e^{X_i(t)}, \quad t \in [0, T],$$

where X_i is the log-price process

$$X_i(t) := \log S_i(t) = X_i(0) + \int_0^t \left(\mu_i(s) - \frac{Y_i^2(s)}{2} \right) ds + \int_0^t Y_i(s) dB_i^S(s).$$

Then, we need a new chain rule. In fact, typically chain rules require

- either all partial derivatives $\frac{\partial f}{\partial x_j}$ to be bounded or Lipschitz (which is not the case for such functions as $f(x) = x^2$ or $f(x) = e^x$)
- or the law of η to be absolutely continuous (which we have not established yet for the log-price $X_i(t)$).

Corollary. For any $i = 1, ..., d$ and $t \in [0, T]$, $X_i(t) \in \mathbb{D}^{1,2}$ and its Malliavin derivative $DX_i(t) = (D^1 X_i(t), ..., D^{2d} X_i(t))$ has the form

$$D_u^j X_i(t) = \left(-\int_0^t Y_i(s) D_u^j Y_i(s) ds + \ell_{i,j} Y_i(u) + \int_0^t D_u^j Y_i(s) dB_i^S(s) \right) \mathbf{1}_{[0,t]}(u).$$

Naturally, we then have $D_u^j S_i(t) = S_i(t) D_u^j X_i(t) \mathbf{1}_{[0,t]}(u)$.

Higher order Malliavin differentiability. In principle, it is intuitively clear how the second order derivative should look like:

$$
\begin{aligned}
D_r D_s Y(t) &= D_r \int_s^t \mathcal{K}(u,s) b'_y(u, Y(u)) \exp\left\{\int_u^t b'_y(v, Y(v)) dv\right\} du \\
&= \int_s^t \mathcal{K}(u,s) D_r \left[b'_y(u, Y(u)) \exp\left\{\int_u^t b'_y(v, Y(v)) dv\right\}\right] du \\
&= \int_s^t \mathcal{K}(u,s) \exp\left\{\int_u^t b'_y(v, Y(v)) dv\right\} D_r \left[b'_y(u, Y(u))\right] du \\
&\quad + \int_s^t \mathcal{K}(u,s) b'_y(u, Y(u)) D_r \left[\exp\left\{\int_u^t b'_y(v, Y(v)) dv\right\}\right] du \\
&= \int_s^t \mathcal{K}(u,s) b''_{yy}(u, Y(u)) \exp\left\{\int_u^t b'_y(v, Y(v)) dv\right\} D_r \left[Y(u)\right] du \\
&\quad + \int_s^t \mathcal{K}(u,s) b'_y(u, Y(u)) \exp\left\{\int_u^t b'_y(v, Y(v)) dv\right\} \int_u^t b''_{yy}(v, Y(v)) D_r[Y(v)] dv du.
\end{aligned}
$$

However, the computations above are far from straightforward to be justified.

- the functions $y \mapsto b'_y(t,y)$ and $y \mapsto b''_{yy}(t,y)$ demonstrate explosive behaviour as $y \to \varphi(t)+$ and $y \to \psi(t)-$ for any $t \in [0,T]$. This makes it impossible to use the classical Malliavin chain rules requiring boundedness of the derivative or demanding the Lipschitz condition.
- In order to overcome this issue, we have to use some special properties of the volatility process established in earlier work and tailor a version of the Malliavin chain rule specifically for our needs.

Then we have to establish the Malliavin chain rule for the random variables $b'_y(t, Y(t))$ and $\exp\left\{\int_u^t b'_y(v, Y(v))dv\right\}$.

Proposition: chain rule applied. For any $0 \le u \le t \le T$ and $p > 1$,

1) $b'_y(t, Y(t)) \in \mathbb{D}^{1,p}$ with

$$D_s\left[b'_y(t, Y(t))\right] = b''_{yy}(t, Y(t))D_sY(t),$$

2) $\exp\left\{\int_u^t b'_y(v, Y(v))dv\right\} \in \mathbb{D}^{1,p}$ with

$$D_s\left[\exp\left\{\int_u^t b'_y(v, Y(v))dv\right\}\right] = \exp\left\{\int_u^t b'_y(v, Y(v))dv\right\}\int_u^t b''_{yy}(v, Y(v))D_sY(v)dv.$$

Theorem. For any $t \in [0, T]$ and $p \ge 2$,

1) $Y(t) \in \mathbb{D}^{2,p}$,

2) with probability 1 and for a.a. $r, s \in [0, T]$,

$$D_rD_sY(t) = \int_s^t \mathcal{K}(u, s)F_1(t, u)\left(\int_u^t b''_{yy}(v, Y(v))D_rY(v)dv\right)du + \int_s^t \mathcal{K}(u, s)F_2(t, u)D_rY(u)du,$$

where

$$F_1(t, u) := b'_y(u, Y(u))\exp\left\{\int_u^t b'_y(v, Y(v))dv\right\},$$

$$F_2(t, u) := b''_{yy}(u, Y(u))\exp\left\{\int_u^t b'_y(v, Y(v))dv\right\}.$$

3. On the power law

To study of the power law of the implied volatility surface $\widehat{\sigma}(\tau, \kappa)$:

$$\left| \frac{\partial \widehat{\sigma}}{\partial \kappa}(\tau, \kappa) \right|_{\kappa=0} = O(\tau^{-\frac{1}{2}+H}), \quad \tau \to 0.$$

we use the criterion by Alòs, León, Vives (2007):

Theorem. Consider a risk-free log-price

$$X(t) = x_0 + rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)\left(\rho dB_1(s) + \sqrt{1-\rho^2}dB_2(s)\right),$$

where B_1, B_2 are two independent Brownian motions, $x_0 \in \mathbb{R}$ is a deterministic initial value, r is an instantaneous interest rate, $\rho \in (-1, 1)$ and $\sigma = \{\sigma(t),\ t \in [0, T]\}$ is a square-integrable stochastic process with right-continuous trajectories adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t,\ t \in [0, T]\}$ generated by B_1. Assume that

- (H1) $\sigma \in \mathbb{L}^{2,4}$ with respect to B_1;
- (H2) there exists a constant $\varphi_* > 0$ such that, with probability 1, $\sigma(t) > \varphi_*$ for all $t \in [0, T]$;
- (H3) σ has a.s. right-continuous trajectories;

(H4) there exists a constant $H \in \left(0, \frac{1}{2}\right)$ such that, with probability 1, for any $0 < s < t < T$,

$$\mathbb{E}\left[(D_s\sigma(t))^2\right] \leq \frac{C}{(t-s)^{1-2H}},$$
$$\mathbb{E}\left[(D_r D_s\sigma(t))^2\right] \leq C\left(\frac{t-r}{t-s}\right)^{1-2H},$$

where $C > 0$ is some constant;

(H5) $\sup_{r,s,t\in[0,\tau]} \mathbb{E}\left[(\sigma(s)\sigma(t) - \sigma^2(r))^2\right] \to 0$ when $\tau \to 0+$.

Finally, assume that there exists a constant $K_\sigma > 0$ such that, with probability 1,

$$\frac{1}{\tau^{\frac{3}{2}+H}} \int_0^\tau \int_s^\tau \mathbb{E}\left[D_s\sigma(t)\right] dtds - K_\sigma \to 0, \quad \tau \to 0+. \tag{H6}$$

Then, with probability 1,

$$\lim_{\tau\to 0} \tau^{\frac{1}{2}-H} \left.\frac{\partial\widehat{\sigma}_{\text{log-price}}}{\partial x}(\tau, x)\right|_{x=\log\frac{K}{e^{r\tau}}} = -\frac{\rho}{\sigma(0)} K_\sigma.$$

RK: Our model naturally satisfied several of the assumptions above:

- assumption (H1) by the results on the moments above.
- assumption (H2) with $\varphi^* := \min_{t\in[0,T]} \varphi(t) > 0$;
- assumption (H3) since Y is continuous a.s.;

The verification of hypotheses (H4), (H5), and (H6) is here below.

Proposition. Under the assumptions of the model. Then with probability 1, (H5) is easy to prove,

$$\sup_{r,s,t\in[0,\tau]} \mathbb{E}\left[(Y(s)Y(t) - Y^2(r))^2\right] \to 0, \quad \tau \to 0.$$

Also, if $H \in \left(\frac{1}{6}, \frac{1}{2}\right)$ and the Volterra kernel $\mathcal{K}$ be such that for any $0 \leq s < t \leq T$

$$|\mathcal{K}(t,s)| \leq C|t-s|^{-\frac{1}{2}+H}$$

for some constant $C > 0$. Then the hypothesis (H4) of the criteria for the power law holds for the volatility process $\sigma = Y$. Furthermore, if the Volterra kernel $\mathcal{K}$ be such that

$$\frac{1}{\tau^{\frac{3}{2}+H}} \int_0^\tau \int_s^\tau \mathcal{K}(t,s)dtds \to K_Y, \quad \tau \to 0+,$$

where K_Y is some finite constant. Then, with probability 1, we have (H6):

$$\frac{1}{\tau^{\frac{3}{2}+H}} \int_0^\tau \int_s^\tau \mathbb{E}\left[D_s Y(t)\right] dtds - K_Y \to 0, \quad \tau \to 0+.$$

Theorem: Power law in SVV models.

Under the assumption of the model with $H \in \left(\frac{1}{6}, \frac{1}{2}\right)$. Assume also that $\rho \neq 0$ and that the Volterra kernel $\mathcal{K}$ is such that for any $0 \leq s < t \leq T$

$$|\mathcal{K}(t,s)| \leq C|t-s|^{-\frac{1}{2}+H}$$

for some constant $C > 0$ and (23) holds with $\rho K_Y < 0$. Then the SVV model reproduces the power law of the at-the-money implied volatility skew with the correct sign.

Example. Let $\frac{1}{6} < H_0 < H_1 < ... < H_n < 1$ be such that $H_0 < \frac{1}{2}$ and $\alpha_k > 0$, $k = 0, ..., n$. Then the kernel

$$\mathcal{K}(t,s) = \left(\sum_{k=0}^{n} \alpha_k (t-s)^{H_k - \frac{1}{2}}\right) \mathbf{1}_{s<t}$$

satisfies the assumptions of Theorem, so the corresponding SVV model generates power law with $H = H_0$ provided that $\rho < 0$.

RK. The condition $H > \frac{1}{6}$ is coherent with the recent empirical estimate $H \approx 0.19$ for the SPX implied volatility obtained in Delemotte, deMarco, Segonne (2023).

Simulation implied volatility surfaces

For simulations, we take the SVV model of the form

$$S(t) = 1 + \int_0^t Y(s)S(s)\left(\sqrt{0.75}dB_1(s) - 0.5dB_2(s)\right),$$

$$Y(t) = 0.5 + \int_0^t \left(\frac{0.005}{(Y(s)-0.005)^5} - \frac{0.005}{(1-Y(s))^5} + 0.05(0.5 - Y(s))\right) ds$$

$$+ 0.3\int_0^t \mathcal{K}(t,s)dB_2(s),$$

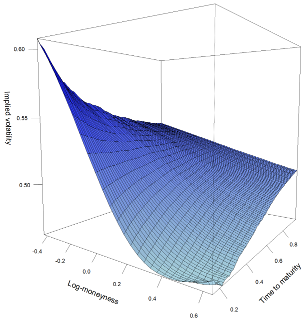
where the choice of $\mathcal{K}$ is varied.

In order to plot the implied volatility surface, we

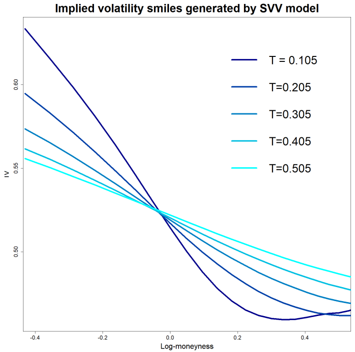
- estimate the call option prices $\mathbb{E}\left[(S(T) - K)_+\right]$ under the model for $T = \frac{n}{200}$, $n = 1, \dots, 200$, and $K = 0.5 + m/100$, $m = 0, \dots, 150$, using the standard Monte Carlo method, i.e. average over 1500000 realisations of the payoff $(S(T) - K)_+$
- calculate the corresponding Black-Scholes implied volatility $(T, \kappa) \mapsto \widehat{\sigma}(T, \kappa)$, where κ denotes the log-moneyness, using a standard procedure. See, e.g., DiNunno, Kubilius, Mishura, Yurchenko-Tytarenko (2023).

Example.

$$\mathcal{K}(t,s) := \frac{1}{\Gamma(0.7)}(t-s)^{-0.3}\mathbf{1}_{s<t}$$



(a) Implied volatility surface



(b) Implied volatility smiles for different maturities

Figure: Implied volatility surface (a) and implied volatility smiles (b) generated by the SVV model with the rough kernel (26). Note that the smile becomes steeper as the time to maturity $T \to 0$, which reproduces a similar effect happening on real markets (for more details, see e.g. Delemotte, de Marco, Segonne (2023), Fouque, Papanicolaou, Sircar, Sølna (2004) or DiNunno, Kubilius, Mishura, Yurchenko-Tytarenko (2023)).

Implied volatility skew generated by SVV model (log-scale)

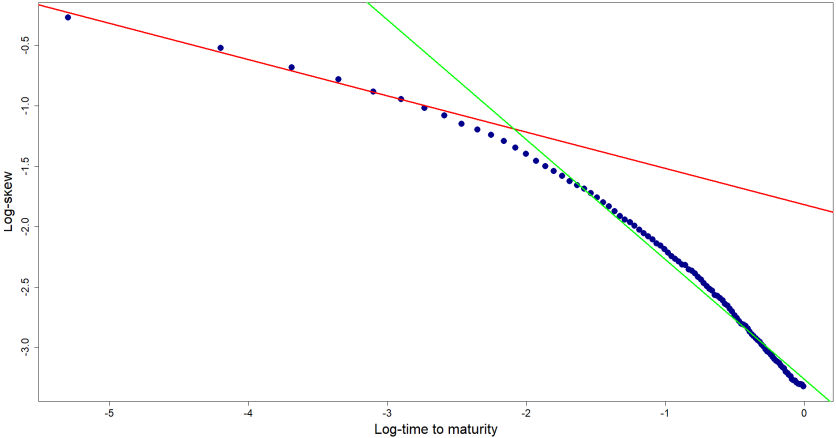
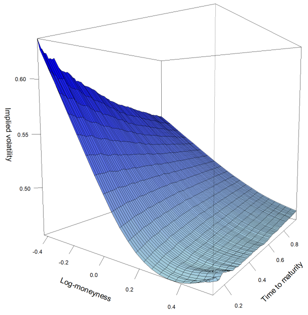


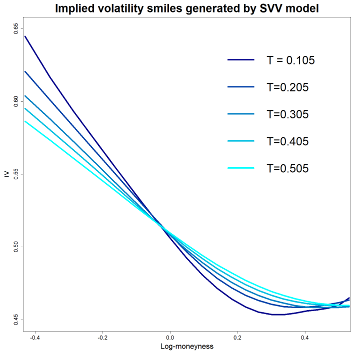
Figure: At-the-money implied volatility skew generated by the SVV model with the rough kernel on a logarithmic scale. The lines depict two linear fits: over the short maturities (red line, the slope is -0.3006433 which is consistent with Theorem 4.8 in DiNunno, Yurchenko-Tytarenko (2023)) and over the long maturities (green line, the slope is -0.9920939). Such a behaviour is consistent with Delemotte, de Marco, Segonne (2023)

Example.

$$\mathcal{K}(t,s) = \left(\frac{\sqrt{0.4}}{\Gamma(0.7)}(t-s)^{-0.3} + \frac{\sqrt{1.8}}{\Gamma(1.4)}(t-s)^{0.4} \right) \mathbf{1}_{s<t}$$



(a) Implied volatility surface



(b) Absolute skews, standard scale

Figure: Implied volatility surface (a) and implied volatility smiles (b) generated by the SVV model with the mixed fractional kernel. Just as in the purely rough case, the absolute at-the-money skew increases for small maturities, which is consistent with the real-life market behaviour. Note that the presence of the term $(t-s)^{0.4}$ in the kernel ensures that the smile "flattens out" for larger maturities much slower than in the purely rough case, which agrees with the results of Funahashi, Kijima (2017)

4. Martingale measures

Theorem The SVV market model is arbitrage-free and incomplete with the set of martingale measure given by $d\mathbb{Q} := M(T)d\mathbb{P}$, where

$$M(t) = \mathcal{E}_t\Big\{ - \sum_{j=1}^{d} \int_0^{\cdot} \Big(\sum_{k=1}^{d} \ell_{j,k}^{(-1)} \frac{\widetilde{\mu}_k(s)}{Y_k(s)} \Big) dW_j(s) + \sum_{l=d+1}^{2d} \int_0^{\cdot} \xi_l(s) dW_l(s) \Big\},$$

$(\widetilde{\mu}_i := \mu_i - \nu)$ is a *uniformly integrable martingale* for any predictable processes ξ_l satisfying the Novikov condition

$$\mathbb{E}\Big[\exp\Big\{ \frac{1}{2} \sum_{l=d+1}^{2d} \int_0^T \xi_l^2(t) dt \Big\} \Big] < \infty.$$

Here above $\big(\ell_{j,k}^{(-1)}\big)_{j,k=1}^{d}$ are the elements of $\mathcal{L}_{11}^{-1}$:

$$\begin{pmatrix} B^S(t) \\ B^Y(t) \end{pmatrix} = \mathcal{L}\, W(t) \qquad \mathcal{L}\,\mathcal{L}^T = \Sigma \qquad \mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$$

Proof. We show that $M(T)$ is a positive density associated with the local $\mathbb{P}$-martingales M such that all $M\widetilde{S}_i$ become local martingales by direct computations. Then we prove that $\mathbb{Q}$ is a probability measure, or equivalently, that M is a uniformly integrable martingale. For this *the boundedness of* Y is crucial.

5. Quadratic hedging

The quadratic hedging problem is to find the optimal $\hat{u}$ such that :

$$J(\hat{u}) = \inf_u J(u) := \inf_u \mathbb{E}\Big[\Big(F - EF - \int_0^T u(s)dX(s)\Big)^2\Big],$$

where the financial claim $F = F(X(T)) \in L^2(\mathbb{P})$ and the infimum is taken over all $\mathbb{F}$-adapted stochastic processes $u \in L^2(\mathbb{P} \times [X])$. Here X is the discounted dynamics and we consider $\mathbb{P} = \mathbb{Q}$.

The solution $\hat{u}$ is stated in terms of the non-anticipating (NA) derivative[6], which is the $L^2(\mathbb{P} \times [X])$-limit

$$\hat{u} = \mathfrak{D}\xi := \lim_{|\pi_M| \to 0} u_{\pi_M}$$

for monotone partitions $\{\pi_M,\ M \geq 1\}$, such that $|\pi_M| \to 0$ as $M \to \infty$, where

$$u_\pi := \sum_{k=0}^{n-1} u_{\pi,k} \mathbf{1}_{(t_k, t_{k+1}]}, \qquad u_{\pi,k} := \mathbb{E}\Big[\xi \frac{X(t_{k+1}) - X(t_k)}{\mathbb{E}\big[(X(t_{k+1}) - X(t_k))^2 \mid \mathcal{G}_{t_k}\big]} \Big| \mathcal{G}_{t_k}\Big]$$

[6] Di Nunno (2002)

About NA-derivatives

Definition. The NA-derivative of ξ wrt X is the $L^2(\mathbb{P} \times [X])$-limit

$$\mathfrak{D}\xi := \lim_{|\pi_M| \to 0} u_{\pi_M}$$

for a monotone partitions $\{\pi_M, \ M \geq 1\}$, such that $|\pi_M| \to 0$ as $M \to \infty$.

Theorem. The NA-derivative $\mathfrak{D}\xi$ is well-defined. Moreover, any $\xi \in L^2(\mathbb{P})$ admits a unique representation of the form

$$\xi = \xi_0 + \int_0^T \mathfrak{D}\xi(s) dX(s),$$

where $\xi_0 \in L^2(\mathbb{P})$ is such that $\mathfrak{D}\xi_0 = 0$ and $\mathbb{E}\left[\xi_0 \int_0^T \mathfrak{D}\xi(s) dX(s)\right] = 0$.

Remark:

- The NA-derivative is the dual of the Itô integral.
- For a payoff function $F \in L^2(\mathbb{P})$ the NA-derivative $u = \mathfrak{D}F$ of F wrt X is indeed provides an explicit representation of the optimal mean-square hedge.
- The corresponding pre-limit sum $u_\pi := \sum_{k=0}^{n-1} u_{\pi,k} \mathbf{1}_{(t_k, t_{k+1}]}$, is the $L^2(\mathbb{P} \times [X])$-orthogonal projection of F onto the subspace generated by stochastic integrals of simple processes.
- Note that admissible portfolios in real markets are exactly of this type since there is no technical possibility of real "continuous" trading.

About the payoffs

We deal with European options with payoffs $F := F(X(T))$ covering both classical contracts and exotic ones with discontinuities (eg, digital, supershare, binary, truncated payoff options) $F \colon \mathbb{R}_+ \to \mathbb{R}$:

$$F = F_1 + F_2,$$

where

(i) F_1 is globally Lipschitz, i.e. there exists $C > 0$ such that
$|F_1(t) - F_1(s)| \leq C|t - s|, \quad s, t \geq 0$;

(ii) $F_2 \colon \mathbb{R}_+ \to \mathbb{R}$ is of bounded variation over $\mathbb{R}_+$, i.e.

$$V(F_2) := \lim_{x \to \infty} V_{[0,x]}(F_2) < \infty,$$

where $V_{[0,x]}(F_2) := \sup \sum_{j=1}^{N} |F_2(x_j) - F_2(x_{j-1})|$ and the supremum above is taken over all $N \geq 0$ and all partitions $0 = x_0 < x_1 < ... < x_N = x$.

Computational challenge in hedging

The NA-derivative gives the explicit formula for computation of hedging.
However, the *computation* of the conditional expectations in the NA-derivative pre-limit is a challenging task, that becomes even more complicated in view of the Volterra noise Z, since it is not Markovian.

Natural idea: Find Markovian approximations.

OK, BUT control the approximating error as well as much as the theoretical stability, eg the approximants need to be well defined dynamics.

- approximate kernel
- approximate volatility model
- approximate prices
- approximate payoff
- approximate minimizer in the approximated hedging problem
- ? Do we obtain an approximation to our target original hedging strategy?

Approx: Quadratic hedging strategy

The NA-derivatives of interest are:

$$\mathfrak{D}F := \lim_{|\pi|\to 0} u_\pi, \qquad \mathfrak{D}F_m := \lim_{|\pi|\to 0} u_\pi^m,$$

with limits in $L^2(\mathbb{P}\times[X])$ and

$$u_\pi := \sum_{k=0}^{n-1} u_{\pi,k}\mathbf{1}_{(t_k,t_{k+1}]}, \quad u_\pi^m := \sum_{k=0}^{n-1} u_{\pi,k}^m\mathbf{1}_{(t_k,t_{k+1}]},$$

with

$$u_{\pi,k} := \frac{\mathbb{E}\left[F\,\Delta X(t_k) \mid \mathcal{F}_{t_k}\right]}{\mathbb{E}\left[(\Delta X(t_k))^2 \mid \mathcal{F}_{t_k}\right]}, \quad u_{\pi,k}^m := \frac{\mathbb{E}\left[F_m\,\Delta X_m(t_k) \mid \mathcal{F}_{t_k}\right]}{\mathbb{E}\left[(\Delta X_m(t_k))^2 \mid \mathcal{F}_{t_k}\right]}$$

where $\Delta X(t_k) := X(t_{k+1}) - X(t_k)$ and $\Delta X_m(t_k) := X_m(t_{k+1}) - X_m(t_k)$ and also $F := F(X(T))$ and $F_m := F(X_m(T))$.

The study of these approximations we consider

$$\mathcal{K}(t,s) = \mathcal{K}(t-s)\mathbf{1}_{s<t}, \quad \mathcal{K}_m(t,s) = \mathcal{K}_m(t-s)\mathbf{1}_{s<t}, \quad t,s \in [0,T].$$

and $\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])} \to 0$, $m \to \infty$.

Then the following statements hold:

1) There exists a constant $C > 0$ that does not depend on m such that,

$$\mathbb{E}\Big[\int_0^T |u_\pi(s) - u_\pi^m(s)|ds\Big] \leq Cn\sqrt{|\pi|}\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^{\frac{1}{5}}.$$

2) If the partitions are such that $n\sqrt{|\pi|}\|\mathcal{K} - \mathcal{K}_{m_n}\|_{L^2([0,T])}^{\frac{1}{5}} \to 0$, $n \to \infty$, then

$$\mathbb{E}\Big[\int_0^T |\mathfrak{D}F(s) - u_\pi^{m_n}(s)|ds\Big] \to 0, \quad n \to \infty.$$

Remark: The exponent $\frac{1}{5}$ appears because of the estimate corresponding to the discontinuous component F_2.

Remark: If $F_2 \equiv 0$, then the results read differently:

1) There exists a constant $C > 0$ that does not depend on m such that

$$\mathbb{E}\Big[\int_0^T |u_\pi(s) - u_\pi^m(s)|ds\Big] \leq Cn\sqrt{|\pi|}\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])},$$

2) Also for $n\sqrt{|\pi|}\|\mathcal{K} - \mathcal{K}_{m_n}\|_{L^2([0,T])} \to 0, \quad n \to \infty$, we have

$$\mathbb{E}\Big[\int_0^T |\mathfrak{D}F(s) - u_\pi^{m_n}(s)|ds\Big] \to 0, \quad n \to \infty.$$

In practice... we consider the kernel approximants:

$$\mathcal{K}_m(t,s) = \sum_{i=1}^{m} e_{m,i}(t) f_{m,i}(s), \quad 0 \le s \le t \le T,$$

where $e_{m,i}$ and $f_{m,i}$ are L^2 functions, so that the $(m+2)$-dimensional process $(X_m, Y_m, U_{m,1}, ..., U_{m,m})$ given by

$$X_m(t) = X(0) + \int_0^t Y_m(s) X_m(s) \left(\rho dB_1(s) + \sqrt{1-\rho^2} dB_2(s) \right),$$

$$Y_m(t) = Y(0) + \int_0^t b(s, Y_m(s)) ds + \sum_{i=1}^{m} e_{m,i}(t) U_{m,i}(t),$$

$$U_{m,1}(t) = \int_0^t f_{m,1}(s) dB_1(s), \ ..., \ U_{m,m}(t) = \int_0^t f_{m,m}(s) dB_1(s),$$

is Markovian wrt the filtration $\mathbb{F} = \{\mathcal{F}_t, \ t \in [0, T]\}$.

Remark: Similar idea was already used in Carmona, Coutin (1998), for the RL-fBm, in Abi Jaber, El Euch (2019) for the rough Heston, and in Baurle, Desmettre (2020) for the fractionally integrated standard Heston models, as well as in Abi Jaber, Miller, Pham (2021) in the context of optimal control in general stochastic Volterra systems.

Rough fractional kernels. Let $H \in \left(0, \frac{1}{2}\right)$ and

$$\mathcal{K}(t,s) = \mathcal{K}(t-s)\mathbf{1}_{s<t} = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)}\mathbf{1}_{s<t},$$

i.e. $Z(t) = \int_0^t \mathcal{K}(t,s)dB_1(s)$ is a RL-fBm. Then the approximants are:

$$\mathcal{K}_m(t-s) = \sum_{i=1}^{m} \sigma_{m,i} e^{-\alpha_{m,i}(t-s)}, \quad 0 \le s < t \le T, \quad m \ge 1.$$

Defined as follows: Let μ be a measure on $\mathbb{R}_+$ defined as ($i = 1, ..., m$)

$$\sigma_{m,i} := \int_{\tau_{m,i-1}}^{\tau_{m,i}} \mu(d\alpha), \quad \alpha_{m,i} := \frac{1}{\sigma_{m,i}} \int_{\tau_{m,i-1}}^{\tau_{m,i}} \alpha\mu(d\alpha), \quad \mu(d\alpha) := \frac{\alpha^{-H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-H\right)} d\alpha,$$

where $0 = \tau_{m,0} < \tau_{m,1} < ... < \tau_{m,m}$ is such that

$$\tau_{m,m} \to \infty, \quad \sum_{i=1}^{m} \int_{\tau_{m,i-1}}^{\tau_{m,i}} (\alpha_{m,i} - \alpha)^2 \mu(d\alpha) \to 0, \quad m \to \infty.$$

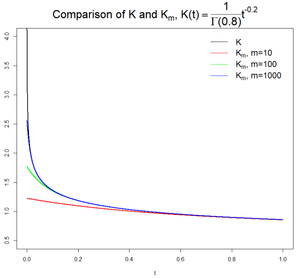
In this case, for uniform partitions, the approx hedging strategies are such that:

1) If $F = F_1 + F_2$ and $m_n := n^\alpha$ for $\alpha > \frac{25}{8H}$, then

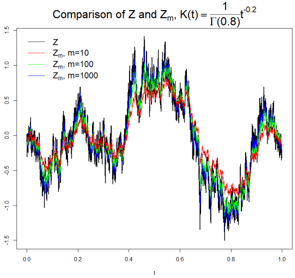
$$\mathbb{E}\left[\int_0^T |u_\pi(s) - u_\pi^{m_n}(s)| ds\right] \leq C n^{-\frac{4\alpha H}{25} + \frac{1}{2}} \to 0, \quad n \to \infty.$$

2) If $F = F_1$ is globally Lipschitz and $m_n := n^\alpha$ for $\alpha > \frac{5}{8H}$, then

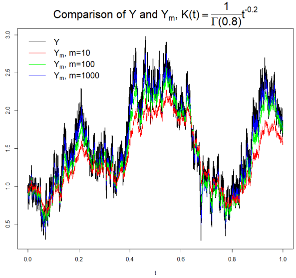
$$\mathbb{E}\left[\int_0^T |u_\pi(s) - u_\pi^{m_n}(s)| ds\right] \leq C n^{-\frac{4\alpha H}{5} + \frac{1}{2}} \to 0, \quad n \to \infty.$$



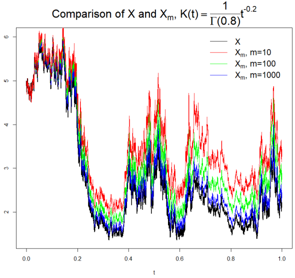
(a) Kernels $\mathcal{K}$ and $\mathcal{K}_m$



(b) Volterra noises Z and Z_m



(c) Volatility processes Y and Y_m



(d) Price processes X and X_m

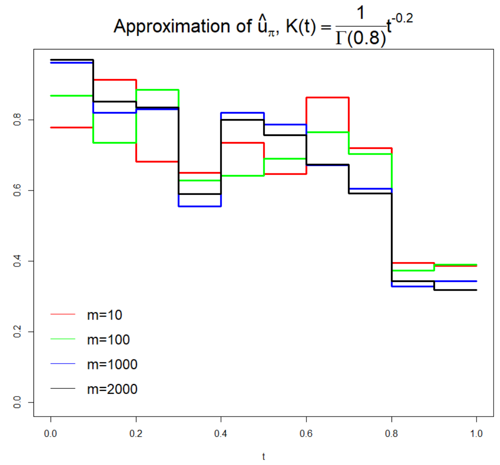


Figure: Hedging strategy for European call with strike 4, estimated for the path above for $m = 10$ (red), $m = 100$ (green), $m = 1000$ (blue) and $m = 2000$ (black). The corresponding partition is $k/10$, $k = 0, 1, \dots, 10$. The figure also illustrates slower rate of convergence in comparison to the Hölder kernel case: the black and blue lines are close to each other but the red and green lines (corresponding to relatively low values of m) differ substantially. Computation time: 24162 seconds for $m = 10$, 25348 seconds for $m = 100$, 38530 seconds for $m = 1000$ and 42431 seconds for $m = 2000$.

Hölder continuous kernels. The fractional kernel

$$\mathcal{K}(t-s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)}, \quad 0 \le s \le t \le T,$$

with $H \in \left(\frac{1}{2}, 1\right)$ is Hölder continuous of order $H - \frac{1}{2}$. The corresponding RL-fBm is λ-Hölder continuous for all $\lambda \in (0, H)$. The approximants are given by the Bernstein polynomial of order m defined as

$$\begin{aligned}
\mathcal{K}_m(t) &= \frac{1}{T^m} \sum_{i=0}^{m} \mathcal{K}\left(\frac{Ti}{m}\right) \binom{m}{i} t^i (T-t)^{m-i} \\
&= \sum_{i=0}^{m} \left(\frac{1}{T^i} \sum_{j=0}^{i} (-1)^{i-j} \mathcal{K}\left(\frac{Tj}{m}\right) \binom{m}{j} \binom{m-j}{i-j} \right) t^i \\
&= \sum_{i=0}^{m} \kappa_{m,i} t^i.
\end{aligned}$$

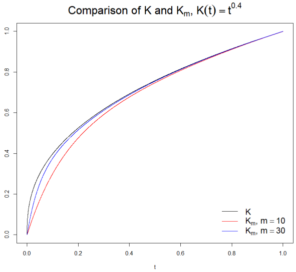
In this case, for uniform partitions, we have

1) If $F = F_1 + F_2$ and $m_n := n^\alpha$ for $\alpha > \frac{5}{H}$, then

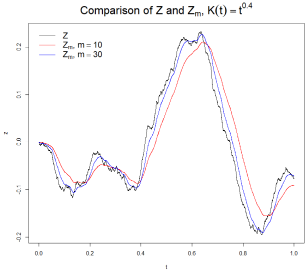
$$\mathbb{E}\left[\int_0^T |u_\pi(s) - u_\pi^{m_n}(s)|ds\right] \leq Cn^{-\frac{\alpha H}{10}+\frac{1}{2}} \to 0, \quad n \to \infty.$$

2) If $F = F_1$ is globally Lipschitz and $m_n := n^\alpha$ for $\alpha > \frac{1}{H}$, then

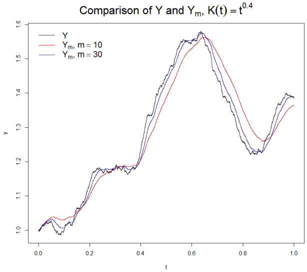
$$\mathbb{E}\left[\int_0^T |u_\pi(s) - u_\pi^{m_n}(s)|ds\right] \leq Cn^{-\frac{\alpha H}{2}+\frac{1}{2}} \to 0, \quad n \to \infty.$$



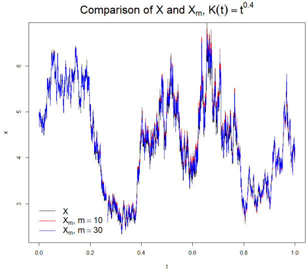
(a) Kernels $\mathcal{K}$ and $\mathcal{K}_m$



(b) Volterra noises Z and Z_m



(c) Volatility processes Y and Y_m



(d) Price processes X and X_m

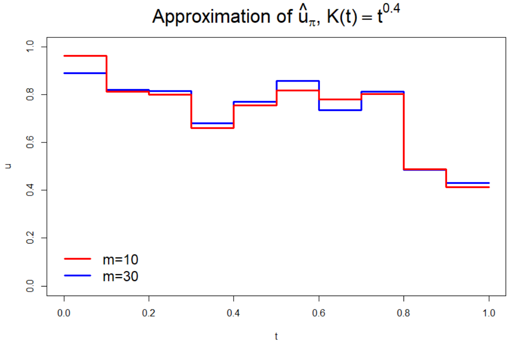


Figure: Hedging strategy for European call option with strike 4, estimated for the path above for $m = 10$ (red) and $m = 30$ (blue). The corresponding partition is $k/10$, $k = 0, 1, \dots, 10$. Simulating the red line took 63231 seconds whereas the blue line – 73410 seconds.

THANK YOU FOR YOUR ATTENTION

Presentation based on:
Di Nunno, G., Yurchenko-Tytarenko, A.: Sandwiched Volterra Volatility model: Markovian approximations and hedging. To appear in Finance and Stochastics, arXiv:2209.13054

And also reference to:
Di Nunno, G., Mishura, Y., and Yurchenko-Tytarenko, A. Option pricing in Volterra Sandwiched Volatility model. SIAM J. Financial Math. 15(3), 2024.
Di Nunno, G., Yurchenko-Tytarenko, A.: Power law of Sandwiched Volterra Volatility models. Modern Stoch. Theory Appl.(2024), 1-26, DOI 10.15559/24-VMSTA246
Di Nunno, G., Mishura, Y., and Yurchenko-Tytarenko, A. Sandwiched SDEs with unbounded drift driven by Holder noises. Advances in Applied Probability 55 (2023).
Di Nunno, G., Mishura, Y., and Yurchenko-Tytarenko, A. Drift-implicit Euler scheme for sandwiched processes driven by Holder noises. Numerical Algorithms (2022).