A Stochastic Stefan Problem With Mushy Region and Turbulent Transport Noise

Ioana CIOTIR
INSA Rouen, France

(joint work with **Franco FLANDOLI** from Scuola Normale Superiore di Pisa, Italy and **Dan GOREAC** from Laval University, Canada)

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The organization of the talk

The organization of the talk is the following.

- Construction of the model
- The rigorous construction of the noise
- Assumptions
- The mathematical model
- Definition of the solution and result
- Sketch of the proof
- Scaling limit of stochastic PDE with turbulent transport

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Construction of the model

Consider a free boundary problem describing the melting/solidification process in a turbulent fluid (a two-phase Stefan problem on a bounded open domain with smooth boundary $\mathcal{O} \subset \mathbb{R}^d$).

$$\begin{cases} C_1 \dfrac{\partial \theta}{\partial t} - div\left(k_1 \nabla \theta\right) + u \cdot \nabla \eta\left(\theta\right) = F, & \text{if } \theta < 0, \\ C_2 \dfrac{\partial \theta}{\partial t} - div\left(k_2 \nabla \theta\right) + u \cdot \nabla \eta\left(\theta\right) = F, & \text{if } \theta > 0, \\ \left(k_2 \nabla \theta^+ - k_1 \nabla \theta^-\right) \cdot N_\xi^- = l \cdot N_t, & \text{on } \{\theta = 0\}, \\ \theta^+ = \theta^- = 0, & \text{on } \{\theta = 0\}, \\ \theta\left(0, \xi\right) = \theta_0\left(\xi\right), & \text{in } \mathcal{O}, \\ \theta\left(t, \xi\right) = 0, & \text{on } \partial\mathcal{O} \times \left(0, T\right), \end{cases} \tag{1}$$

where θ^+, resp. θ^- are the right, resp. left limits of the free boundary situated between the solid and the liquid phase, $N\left(t, \xi\right)$ is the unit normal to the interface and l is assumed to be the latent heat.

Construction of the model

We denote by

- k_1 and k_2 the thermal conductivity of the solid and liquid phases
- C_1 and C_2 the specific heat of the two phases
- the function $\eta : \mathbb{R} \to \mathbb{R}$ is assumed to be a smooth function which vanishes in the solid phase, and such that $\eta(0) = 0$.
  We assume that $|\eta'(r)| \leq L$, for $\forall r \in \mathbb{R}$.

From the physical point of view, it is coherent to strengthen the null-behavior, and further assume that

$$\eta(\theta) = 0, \text{ for } 0 < \theta < \varepsilon.$$

Since the term $u \cdot \nabla \eta(\theta)$ is meant to model the turbulence present in the liquid phase, the physical interpretation of η is that the solid phase is not allowed to move and neither is a small liquid region close to the boundary.

Construction of the model

The convection term has the form

$$u \cdot \nabla \eta (\theta) . \tag{2}$$

Concerning the *velocity* of the fluid, it can be seen in several ways.

A deterministic approach consist in taking u as the solution of a Navier Stockes equation

- Barbu, V., Ciotir, I., Danaila, I., (2021) Existence and uniqueness of solution to the two-phase Stefan problem with convection, Applied Mathematics and Optimization, 84(2).

Construction of the model

A stochastic reasonable model for the turbulent fluid has the form

$$u(t,\xi) = \sum_{k=1}^{\infty} \alpha_k \sigma_k(\xi) \frac{d\beta_k(t)}{dt},$$

where

- $\{\alpha_k\}_k$ is a sequence of positive constants conveniently chosen;
- $\{\sigma_k\}_k$ is a sequence of divergence-free smooth vector fields whose properties will be defined later on, and
- $\{\beta_k\}_k$ is a sequence of independent Brownian motions.

Interpreted in the *Stratonovich sense*, the turbulence has the following formulation

$$u(t,\xi) = \sum_{k=1}^{\infty} \alpha_k \sigma_k(\xi) \circ d\beta_k(t).$$

Construction of the model

We obtain the following explicit formulation for the turbulence

$$u \cdot \nabla \eta\left(\theta\right) = \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla \eta\left(\theta\right) \circ d\beta_k. \tag{3}$$

The heuristic idea is that turbulence can appear in the liquid region of the phase change problem due to the difference in temperature between the two phases.

This type of noise has been introduced and intensively studied during the recent years. The reader is invited to refer to

- F. Flandoli, L. Galeati, D. Luo, *Scaling limit of 2D Euler equations with transport noise to the deterministic Navier-Stokes equations*, Journal of Evolution Equations, Volume 21, pages 567-600, 2021.
- F. Flandoli and E. Luongo. *Stochastic Partial Differential Equa- tions in Fluid Mechanics*, volume 2328. Springer Nature, 2023.

among other references.

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The rigorous construction of the noise

We make a rigorous construction of the turbulence term.

- Let $\{e_k\}_k$ be a complete orthonormal basis in $L^2(\mathcal{O})$ formed by the eigenfunctions of the Dirichlet homogeneous Laplace operator on $\mathcal{O}$, with $\{\lambda_k\}_k$ being the corresponding eigenvalues, i.e.,

$$-\Delta e_k = \lambda_k e_k, \quad \forall k \in \mathbb{N}^*.$$

- We consider $\{\mu_k\}_k$ to be a sequence of divergence-free vectors belonging to $(C^\infty(\mathcal{O}))^d$, and such that

$$div\,(\mu_k e_k) = 0$$

.

- We take $\{\sigma_k\}_k$ a sequence of vectors of the type

$$\sigma_k = \mu_k e_k, \quad \forall k = \overline{1, \infty}.$$

The rigorous construction of the noise

We define the operator $B : H_0^1(\mathcal{O}) \to L_2\left(L^2(\mathcal{O}); L^2(\mathcal{O})\right)$ where

$$\begin{aligned} B(\theta) &: L^2(\mathcal{O}) \to L^2(\mathcal{O}) \\ B(\theta)(\varphi) &= \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla\eta(\theta)(e_k, \varphi)_2 \end{aligned}$$

where $\{\alpha_k\}_k$ is a sequence of real values.
Under the assumptions below, B is well defined from $H_0^1(\mathcal{O})$ to the space of Hilbert-Schmidt operators $L_2\left(L^2(\mathcal{O}); L^2(\mathcal{O})\right)$

$$\begin{aligned} \|B(\theta)\|^2_{L_2(L^2(\mathcal{O});L^2(\mathcal{O}))} &= \sum_{k=1}^{\infty} |\alpha_k \sigma_k \cdot \nabla\eta(\theta)|_2^2 \\ &\leq L^2 C^2 \sum_{k=1}^{\infty} |\alpha_k|^2 |\lambda_k|^2 |\mu_k|^2_{(L^\infty(\mathcal{O}))^d} |\theta|^2_{H_0^1(\mathcal{O})}. \end{aligned}$$

The rigorous construction of the noise

We consider a *cylindrical Wiener process* in $L^2(\mathcal{O})$ constructed with respect to the aforementioned basis

$$dW(t) = \sum_{k=1}^{\infty} e_k d\beta_k(t)$$

where $\{\beta_k\}_k$ is a sequence of independent standard Brownian motions on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$.
The noise term of the equation is meant in the Stratonovich sense, i.e., (16) can be written as

$$B(\theta) \circ dW(t) = \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla \eta(\theta) \circ d\beta_k(t),$$

which means that the noise (16) can be written as

$$u \cdot \nabla \eta(\theta) = B(\theta) \circ dW(t).$$

The rigorous construction of the noise

In order to study the equation, we can transform the Stratonovich integral into a Itô one, whichever is more convenient for differential formulations:

$$\begin{aligned} B(\theta) \circ dW(t) &= \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla \eta(\theta)\, d\beta_k(t) \\ &\quad - \frac{1}{2} \sum_{k=1}^{\infty} \alpha_k^2 div \left[\left(\eta'(\theta)\right)^2 \sigma_k \otimes \sigma_k \nabla \theta \right] dt. \end{aligned}$$

We denote by

$$Q(\xi) = \sum_{k=1}^{\infty} \alpha_k^2 \left(\sigma_k(\xi) \otimes \sigma_k(\xi)\right). \tag{4}$$

The elements α are now considered small enough to guarantee an absolute convergence in the previous expression.

In order to facilitate the reading, we denote the matrix operator $Q = (q_{i,j})_{1 \leq i,j \leq d}$ where each $q_{i,j}$ is a series in k which converges under the assumptions mentioned at the end of the introduction.

The rigorous construction of the noise

In order to study the equation, we can transform the Stratonovich integral into a Itô one, whichever is more convenient for differential formulations:

$$\begin{aligned} B(\theta)\circ dW(t) &= \sum_{k=1}^{\infty}\alpha_k\sigma_k\cdot\nabla\eta(\theta)\,d\beta_k(t) \\ &\quad -\frac{1}{2}\sum_{k=1}^{\infty}\alpha_k^2 div\left[\left(\eta'(\theta)\right)^2\sigma_k\otimes\sigma_k\nabla\theta\right]dt. \end{aligned}$$

On the other hand, we introduce the real function

$$g(x)=\frac{1}{2}\int_0^x\left(\eta'(r)\right)^2 dr,\quad \forall x\in\mathbb{R},$$

The rigorous construction of the noise

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$$\begin{aligned} B(\theta) \circ dW(t) &= \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla \eta(\theta)\, d\beta_k(t) \\ &\quad - \frac{1}{2} \sum_{k=1}^{\infty} \alpha_k^2 div \left[\left(\eta'(\theta) \right)^2 \sigma_k \otimes \sigma_k \nabla \theta \right] dt. \end{aligned}$$

The corresponding Itô integral has the following form

$$B(\theta)\, dW(t) = \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla \eta(\theta)\, d\beta_k(t),$$

followed by a correction term.
We rewrite the noise as

$$B(\theta) \circ dW(t) = B(\theta)\, dW(t) - div \left[Q \nabla g(\theta) \right] dt. \tag{5}$$

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Assumptions

Throughout this paper we shall assume that the elements of the sequence $\{\alpha_k\}_k$ are small enough such that the following series are converging

- For the well-posedness of the operator B from the noise we assume that

$$\sum_{k=1}^{\infty} |\alpha_k|^2 |\lambda_k|^2 |\mu_k|^2_{(L^\infty(\mathcal{O}))^d} \leq C_1 < \infty, \tag{6}$$

  for some constant C_1.

- For the well-posedness of the operator $Q = (q_{ij})_{1 \leq i,j \leq d}$ (with each q_{ij} a series in k) which appears in the Itô-Stratonovich correction term, we assume that

$$\sum_{k=1}^{\infty} \alpha_k^2 (\sigma_k(\xi) \otimes \sigma_k(\xi)) \text{ are convergent for almost every } \xi \in \mathcal{O}. \tag{7}$$

Assumptions

Throughout this paper we shall assume that the elements of the sequence $\{\alpha_k\}_k$ are small enough such that the following series are converging

- For the well-posedness of the problem we further assume that

$$\gamma = \max_{i,j=\overline{1,d}} \left\{ |q_{ij}|_\infty + \left| \frac{\partial q_{ij}}{\partial \xi_i} \right|_\infty \right\} < \infty. \quad (8)$$

  and the matrix Q is positively defined, i.e.

$$a^t Q a \geq 0, \quad \forall a \in \mathbb{R}^d,$$

  where $|\cdot|_\infty$ denotes the $L^\infty(\mathcal{O})$ norm.
- We assume that $\widetilde{\gamma}$ and its inverse are smooth and $0 \leq \left(\widetilde{\gamma}^{-1}\right)' \leq 1$.

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The mathematical model

Under the assumptions above we can rigorously write the system (1) as follows

$$\begin{cases} C_1 d\theta - div\left(k_1 \nabla\theta\right) dt - div\left[Q\nabla g\left(\theta\right)\right] dt + B\left(\theta\right) dW\left(t\right) = F, & \theta < 0, \\ C_2 d\theta - div\left(k_2 \nabla\theta\right) dt - div\left[Q\nabla g\left(\theta\right)\right] dt + B\left(\theta\right) dW\left(t\right) = F, & \theta > 0, \\ \left(k_2 \nabla\theta^+ - k_2 \nabla\theta^-\right) \cdot N_\xi^- = l \cdot N_t, & \theta = 0, \\ \theta^+ = \theta^- = 0, & \theta = 0, \\ \theta\left(0,\xi\right) = \theta_0\left(\xi\right), & \mathcal{O}, \\ \theta\left(t,\xi\right) = 0, & \partial\mathcal{O}. \end{cases} \tag{9}$$

The mathematical model

We shall first write it as a nonlinear multi-valued problem of monotone type

$$\begin{cases} d\gamma\left(\theta\right) - div\left(k\left(\theta\right)\nabla\theta\right)dt - div\left(Q\nabla g\left(\theta\right)\right)dt \\ \qquad\qquad\qquad\qquad +B\left(\theta\right)dW\left(t\right) = F, & \mathcal{O}\times\left(0,T\right), \\ \theta\left(0,\xi\right) = \theta_0\left(\xi\right), & \mathcal{O}, \\ \theta\left(t,\xi\right) = 0, & \partial\mathcal{O}\times\left(0,T\right). \end{cases} \tag{10}$$

where $\gamma\left(r\right) = C\left(r\right) + l \times H\left(r\right)$ with

$$C\left(r\right) = \begin{cases} C_1 r, & r \leq 0, \\ C_2 r, & r > 0. \end{cases}$$

The function H is a Heaviside contribution. Furthermore, we consider

$$k\left(r\right) = \begin{cases} k_1, & r \leq 0, \\ k_2, & r > 0. \end{cases}$$

At this point, by formally using the change of variable $\gamma(\theta) = X_0$, we can rewrite the equation above as

$$\begin{cases} dX_0 - \Delta\Psi_0(X_0)\,dt - div\left(Q\nabla g\left(\gamma^{-1}(X_0)\right)\right)dt & \\ \qquad\qquad + B\left(\gamma^{-1}(X_0)\right)dW(t) = F, & \mathcal{O}\times(0,T), \\ X_0(0,\xi) = \gamma(\theta_0)(\xi) \overset{not}{=} x, & \mathcal{O}, \\ X_0(t,\xi) = 0, & \partial\mathcal{O}\times(0,T) \end{cases} \tag{11}$$

where $\Psi_0(r) = \begin{cases} k_1 C_1^{-1} r, & r \leq 0, \\ 0, & r \in (0,l), \\ k_2 C_2^{-1}(r-l), & r \geq l. \end{cases}$

Since the interface between ice and water is not sharp, we assume the presence of a *mushy region* which is mathematically take into account by replacing the Heaviside function with a smoothed one denoted by $\widetilde{H}$.

The mathematical model

The new equation is

$$\begin{cases} dX - \Delta\Psi(X)\,dt - div\left(Q\nabla g\left(\widetilde{\gamma}^{-1}(X)\right)\right)dt \\ \qquad\qquad +B\left(\widetilde{\gamma}^{-1}(X)\right)dW(t) = F, & \mathcal{O}\times(0,T), \\ X(0,\xi) = \widetilde{\gamma}(\theta_0)(\xi) \overset{not}{=} x, & \mathcal{O}, \\ X(t,\xi) = 0, & \partial\mathcal{O}\times(0,T) \end{cases} \tag{12}$$

where Ψ is a smoothening version (obtained, for instance, by reasonable convolution/mollification) in such a way that the function Ψ is null at 0, strictly monotone, which means that there exist a positive constant ψ_0 such that

$$(\Psi(x) - \Psi(y))(x-y) \geq \psi_0 |x-y|^2, \quad \forall x, y \in \mathbb{R}.$$

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Definition of the solution

Definition

Let $x \in L^2(\mathcal{O})$. We say that equation (12) has a weak solution if there exist a filtered reference probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, a sequence of independent $\mathcal{F}_t$ Brownian motions $\{\beta_k\}_k$, and an $H^{-1}(\mathcal{O})-$valued continuous $\mathcal{F}_t-$adapted process X such that $X \in L^2(\Omega \times (0, T) \times \mathcal{O})$, and the following holds true

$$\begin{aligned}
(X(t), e_j)_2 &= (x, e_j)_2 + \int_0^t \int_{\mathcal{O}} F(s) e_j d\xi ds + \int_0^t \int_{\mathcal{O}} \Psi(X(s)) \Delta e_j d\xi ds \\
&\quad \int_0^t \int_{\mathcal{O}} g\left(\widetilde{\gamma}^{-1}(X)\right) div\left[Q \nabla e_j\right] d\xi ds \\
&\quad + \sum_{k=1}^{\infty} \int_0^t \alpha_k \left(\eta\left(\widetilde{\gamma}^{-1}(X)\right), \sigma_k \cdot \nabla e_j\right)_2 d\beta_k(s),
\end{aligned}$$

$\mathbb{P}$-a.s., for all $t \in [0, T]$, and for all $j \in \mathbb{N}^*$, where $\{e_j\}$ is the orthonormal basis in $L^2(\mathcal{O})$ as introduced before.

The main result

We now come to the main result of the paper.

Theorem

For each $x \in L^2(\mathcal{O})$*, there exists a solution to the equation (12), in the sense of the Definition 1, and such that*

$$\Psi(X) \in L^2(\Omega \times (0, T) \times \mathcal{O}).$$

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Sketch of the proof

We consider the Gelfand triple $V \subset H \subset V^*$ where $V = H_0^1(\mathcal{O})$, $H = L^2(\mathcal{O})$ and $V^* = H^{-1}(\mathcal{O})$.
We let $H_n = span\{e_1, e_2, ..., e_n\}$, such that $span\{e_i \,|i \in \mathbb{N}|\}$ is dense in V.
Let $P_n : V^* \to H_n$ be defined by

$$P_n y = \sum_{i=1}^{n} (y, e_i)_H \, e_i, \quad y \in V^*.$$

Clearly, $P_n|_H$ is just the orthogonal projection onto H_n in H.

We take as initial data $X^{(n)}(0) = P_n x$. For (coherence and) notation purposes, we will also employ $x^{(n)} = P_n x$ to denote the initial datum for the approximating solutions.

Sketch of the proof

For each finite $n \in \mathbb{N}$ we consider the following equation in H_n

$$\begin{aligned} dP_nX &= P_n\Delta\Psi\left(P_nX\right)dt + P_n div\left[P_n\left(Q\nabla g\left(\widetilde{\gamma}^{-1}\left(P_nX\right)\right)\right)\right]dt \\ &\quad - P_nB\left(\widetilde{\gamma}^{-1}\left(P_nX\right)\right)dW\left(t\right) \end{aligned}$$

which has a unique strong solution in finite dimension since the operators are Lipschitz-continuous

$$\begin{aligned} dX^{(n)} &= \Delta P_n\Psi\left(X^{(n)}\right)dt + P_n div\left[P_n\left(Q\nabla g\left(\widetilde{\gamma}^{-1}\left(X^{(n)}\right)\right)\right)\right]dt \\ &\quad - \sum_{k=1}^{n} P_n\left(\alpha_k\sigma_k\cdot\nabla\eta\left(\widetilde{\gamma}^{-1}\left(X^{(n)}\right)\right)\right)d\beta_k\left(t\right), \end{aligned}$$

where $P_nX = X^{(n)}$. Local existence of the solution is a classical fact due to the regularity of the coefficients appearing in the equation.

We first apply the Itô formula in $L^2(\mathcal{O})$ and we get

$$\begin{aligned}
& \left\| X^{(n)}(t) \right\|_2^2 + 2\int_0^t \int_{\mathcal{O}} \Psi'\left(X^{(n)}\right)\left|\nabla X^{(n)}\right|^2 d\xi ds \\
& +2\int_0^t \int_{\mathcal{O}} \left(\nabla\left(\left(g\circ\widetilde{\gamma}^{-1}\right)\left(X^{(n)}\right)\right)\right)^t Q\nabla X^{(n)} d\xi ds \\
= \; & \left\| X^{(n)}(0) \right\|_2^2 + \sum_{k=1}^{n}\int_0^t \int_{\mathcal{O}} \left| P_n\left(\alpha_k\sigma_k\cdot\nabla\left(\left(\eta\circ\widetilde{\gamma}^{-1}\right)\left(X^{(n)}\right)\right)\right)\right|^2 d\xi ds \\
& +2\sum_{k=1}^{n}\int_0^t \alpha_k\left(\left(\eta\circ\widetilde{\gamma}^{-1}\right)\left(X^{(n)}\right), \sigma_k\cdot\nabla X^{(n)}\right)_2 d\beta_k(s).
\end{aligned}$$

Keeping in mind that $div\ \sigma_k = 0$, we can see that

$$\left(\left(\eta\circ\widetilde{\gamma}^{-1}\right)\left(X^{(n)}\right), \sigma_k\cdot\nabla X^{(n)}\right)_2 = -\left(\Upsilon\left(X^{(n)}\right), div\ \sigma_k\right)_2 = 0$$

where Υ is a primitive of $\eta\circ\widetilde{\gamma}^{-1}$.

Sketch of the proof

We first apply the Itô formula in $L^2(\mathcal{O})$ and we get

$$\begin{aligned} & \left\| X^{(n)}(t) \right\|_2^2 + 2 \int_0^t \int_{\mathcal{O}} \Psi' \left(X^{(n)} \right) \left| \nabla X^{(n)} \right|^2 d\xi ds \\ & + 2 \int_0^t \int_{\mathcal{O}} \left(\nabla \left(\left(g \circ \widetilde{\gamma}^{-1} \right) \left(X^{(n)} \right) \right) \right)^t Q \nabla X^{(n)} d\xi ds \\ = & \left\| X^{(n)}(0) \right\|_2^2 + \sum_{k=1}^{n} \int_0^t \int_{\mathcal{O}} \left| P_n \left(\alpha_k \sigma_k \cdot \nabla \left(\left(\eta \circ \widetilde{\gamma}^{-1} \right) \left(X^{(n)} \right) \right) \right) \right|^2 d\xi ds \\ & + 2 \sum_{k=1}^{n} \int_0^t \alpha_k \left(\left(\eta \circ \widetilde{\gamma}^{-1} \right) \left(X^{(n)} \right), \sigma_k \cdot \nabla X^{(n)} \right)_2 d\beta_k(s). \end{aligned}$$

Keeping in mind that $div\ \sigma_k = 0$, we can see that

$$\left(\left(\eta \circ \widetilde{\gamma}^{-1} \right) \left(X^{(n)} \right), \sigma_k \cdot \nabla X^{(n)} \right)_2 = - \left(\Upsilon \left(X^{(n)} \right), div\ \sigma_k \right)_2 = 0$$

where Υ is a primitive of $\eta \circ \widetilde{\gamma}^{-1}$.

We obtain

$$\sup_{t\in[0,T]} \left\|X^{(n)}(t)\right\|_2^2 + 2\int_0^t \int_{\mathcal{O}} \Psi'\left(X^{(n)}\right)\left|\nabla X^{(n)}\right|^2 d\xi ds \le \left\|X^{(n)}(0)\right\|_2^2, \tag{13}$$

uniformly for all $\omega \in \Omega$.

We deduce that, for every $p > 1$, and along some subsequence,

$$\begin{aligned} X^{(n)} &\rightharpoonup X \text{ weakly in } L^p\left(0,T;L^2(\mathcal{O})\right), \quad \mathbb{P}\text{-a.s.} \\ X^{(n)} &\rightharpoonup X \text{ weakly in } L^2\left((0,T);H_0^1(\mathcal{O})\right), \quad \mathbb{P}\text{-a.s..} \end{aligned} \tag{14}$$

By recalling that the functions Ψ, g, η and $\widetilde{\gamma}^{-1}$ are Lipschitz-continuous, we also have that

$$\begin{aligned} \Psi\left(X^{(n)}\right) &\rightharpoonup \varkappa \text{ weakly in } L^2\left((0,T);L^2(\mathcal{O})\right), \quad \mathbb{P}\text{-a.s.} \\ g\left(\widetilde{\gamma}^{-1}\left(X^{(n)}\right)\right) &\rightharpoonup \rho \text{ weakly in } L^2\left((0,T);L^2(\mathcal{O})\right), \quad \mathbb{P}\text{-a.s.} \\ \eta\left(\widetilde{\gamma}^{-1}\left(X^{(n)}\right)\right) &\rightharpoonup \zeta \text{ weakly in } L^2\left((0,T);L^2(\mathcal{O})\right), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Sketch of the proof

In order to identify the limits of the nonlinear terms, we need to show the strong convergence of the approximating solutions in $H^{-1}(\mathcal{O})$. To this purpose we introduce the inclusions

$$L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O}) \subset H^{-\beta}(\mathcal{O}),$$

where β is assumed to be large enough.

We intend to use the compactness result in Corollary 5 from

- J. Simon, *Compact sets in the space $L^p(0,T;B)$*. Ann. Mat. Pura Appl. 146, 65-96, 1987.

According to this result, one has

$$L^\infty\left(0,T;L^2(\mathcal{O})\right) \cap W^{\alpha,r}\left(0,T;H^{-\beta}(\mathcal{O})\right) \subset C\left([0,T];H^{-1}(\mathcal{O})\right),$$

with compact inclusion, provided that $\alpha < \frac{1}{2}$, $\beta > 4$, $r \geq 4$ and $\alpha r > 1$.

Sketch of the proof

To this purpose, we rely on the following lemmas.

Lemma

There exists a constant C independent of n such that for $r \geq 4$ and all $0 \leq s \leq t \leq T$ it holds

$$\mathbb{E}\left[\left|\left(X_t^{(n)} - X_s^{(n)}, e_j\right)_2\right|^r\right] \leq C\,|t-s|^{\frac{r}{2}}\left(\lambda_j^r + \|div\left[Q\left(\xi\right)\nabla e_j\right]\|_2^r + \lambda_j^{\frac{r}{2}}\right).$$

The result is proven by estimating each of the terms appearing in the definition of the solution.

Lemma

For $\beta > 4$ and $r \geq 4$, there is a constant C independent of n such that

$$\mathbb{E}\left[\left\|X_t^{(n)} - X_s^{(n)}\right\|_{H^{-\beta}}^r\right] \leq C\,|t-s|^{\frac{r}{2}}.$$

Sketch of the proof

We can now use the previous Lemmas in order to get

$$\int_0^T \int_0^T \frac{\mathbb{E}\left\|X^{(n)}(t) - X^{(n)}(s)\right\|_{H^{-\beta}}^r}{|t-s|^{1+\alpha r}} dtds \leq \int_0^T \int_0^T \frac{C\,|t-s|^{\frac{r}{2}}}{|t-s|^{1+\alpha r}} dtds \leq C,$$

since $1 + \alpha r - \frac{r}{2} < 1$ for $\alpha \in \left(0, \frac{1}{2}\right)$ and $r \geq 4$.
We get that the set

$$K_R = \left\{ f \in C\left(0, T; H^{-1}(\mathcal{O})\right); \|f\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \|f\|_{W^{\alpha,r}\left(0,T;H^{-\beta}(\mathcal{O})\right)} \leq R \right.$$

is a compact set in the space $C\left([0, T]; H^{-1}(\mathcal{O})\right)$.

Sketch of the proof

By using Markov's inequality, we have

$$\begin{aligned}\nu_n\left(K_R^c\right) &= \mathbb{P}\left(X^{(n)} \in K_R^c\right)\\ &= \mathbb{P}\left(\left\|X^{(n)}\right\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \left\|X^{(n)}\right\|_{W^{\alpha,r}\left(0,T;H^{-\beta}(\mathcal{O})\right)} > R\right)\\ &\leq \frac{1}{R}\mathbb{E}\left[\left\|X^{(n)}\right\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \left\|X^{(n)}\right\|_{W^{\alpha,r}\left(0,T;H^{-\beta}(\mathcal{O})\right)}\right] \leq \frac{C}{R} \leq \varepsilon,\end{aligned}$$

for R sufficiently large and we obtain that the family of laws $\{\nu_n\}_n$ is tight in the space $C\left([0, T]; H^{-1}(\mathcal{O})\right)$.

As a consequence, by Skorohod's theorem, there exists a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ endowed with a filtration $\left(\widetilde{\mathcal{F}}_t\right)_{t\in[0,T]}$, a sequence of filtrations $\left(\widetilde{\mathcal{F}}_t^{(n)}\right)_{t\in[0,T]}$, and the stochastic processes $\widetilde{X}^{(n)}$ with the $\left(\widetilde{\mathcal{F}}_t^{(n)}\right)_{t\in[0,T]}$ cylindrical Wiener process $\widetilde{W}^{(n)} = \sum_{k=1}^{\infty} e_k \widetilde{\beta}_t^{k,n}$ and also $\widetilde{X}$ with the $\left(\widetilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ cylindrical Wiener process $\widetilde{W} = \sum_{k=1}^{\infty} e_k \widetilde{\beta}_t^{k}$ on $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$.
Furthermore, the law of $\widetilde{X}^{(n)}$ is the same as the law of $X^{(n)}$, the law of $\widetilde{W}^{(n)}$is the same as the law of $W^{(n)}$and

$$\begin{aligned} \widetilde{X}^{(n)} &\longrightarrow \widetilde{X} \text{ strongly in } C\left([0,T]; H^{-1}(\mathcal{O})\right),\ \widetilde{\mathbb{P}}-a.s. \\ \widetilde{\beta}_t^{k,n} &\longrightarrow \widetilde{\beta}_t^{k} \text{ strongly in } C\left([0,T]; \mathbb{R}\right),\ \widetilde{\mathbb{P}}-a.s.,\ \forall k \geq 0, \end{aligned} \tag{15}$$

as $n \to \infty$.

Sketch of the proof

We consider the limit in the (PDE) weak formulation of approximating solutions, i.e.,

$$\begin{aligned}\left(\widetilde{X}^{(n)}(t), e_j\right)_2 &= \left(x^{(n)}, e_j\right)_2 + \int_0^t \int_{\mathcal{O}} \Psi\left(\widetilde{X}^{(n)}(s)\right) \Delta e_j d\xi ds \\ &+ \int_0^t \int_{\mathcal{O}} g\left(\widetilde{\gamma}^{-1}\left(\widetilde{X}^{(n)}\right)\right) div\left[Q\nabla e_j\right] d\xi ds \\ &+ \sum_{k=1}^{n} \int_0^t \alpha_k \left(\eta\left(\widetilde{\gamma}^{-1}\left(\widetilde{X}^{(n)}\right)\right), \sigma_k \cdot \nabla e_j\right)_2 d\widetilde{\beta}_s^{k,n},\end{aligned}$$

$\widetilde{\mathbb{P}} - a.s.$ and we obtain that

$$\begin{aligned}\left(\widetilde{X}(t), e_j\right)_2 &= (x, e_j)_2 + \int_0^t \int_{\mathcal{O}} \Psi\left(\widetilde{X}(s)\right) \Delta e_j d\xi ds \\ &+ \int_0^t \int_{\mathcal{O}} g\left(\widetilde{\gamma}^{-1}\left(\widetilde{X}\right)\right) div\left[Q(\xi)\nabla e_j\right] d\xi ds \\ &+ \sum_{k=1}^{\infty} \int_0^t \alpha_k \left(\eta\left(\widetilde{\gamma}^{-1}\left(\widetilde{X}\right)\right), \sigma_k \cdot \nabla e_j\right)_2 d\widetilde{\beta}_s^{k}, \ \mathbb{P} - a.s.\end{aligned}$$

The organization of the talk

The organization of the talk is the following.

- Construction of the model
- The rigorous construction of the noise
- Assumptions
- The mathematical model
- Definition of the solution and result
- Sketch of the proof
- Scaling limit of stochastic PDE with turbulent transport

Scaling limit of stochastic PDE

We consider again the equation

$$\begin{cases} dX - \Delta K\left(\gamma^{-1}(X)\right)dt + u\cdot\nabla\eta\left(\gamma^{-1}(X)\right) = F \\ X(0,\xi) = x_0(\xi). \end{cases}$$

For more clearness we shall denote by $\Psi = K\circ\gamma^{-1}$ and $\Gamma = \eta\circ\gamma^{-1}$ and rewrite the equation as follows

$$\begin{cases} dX - \Delta\Psi(X)\,dt + u\cdot\nabla\Gamma(X) = F \\ X(0,\xi) = x_0(\xi). \end{cases}$$

In this setting we assume that

- Ψ and Γ are assumed to be strictly monotone such that $\Psi' \geq \psi_0 > 0$ and $\Gamma' \geq \gamma_0 > 0$ and Lipschitz continuos, and to be null in zero.
- $\widetilde{\gamma}^{-1}$ is assumed to satisfy that $\left(\widetilde{\gamma}^{-1}\right)' < 1$. (This assumption is not restrictive and is necessary in order to apply the existence result for the stochastic equation.)

Scaling limit of stochastic PDE

We study this equation in the two dimensional torus $\Pi^2 = \mathbb{R}^2/\mathbb{Z}^2$ and we consider $\left(H^{s,p}\left(\Pi^2\right), \|\cdot\|_{H^{s,p}}\right)$, $s \in \mathbb{R}$, $p \in (1, \infty)$ a Bessel space of zero mean periodic functions.

The turbulence term $u \cdot \nabla\Gamma\left(X\right)$ we take the Stratonovich interpretation

$$u\left(t, \xi\right) = \sum_{k \in \mathbb{Z}_0^2} \alpha_k \sigma_k \circ d\beta_k$$

and we get

$$u \cdot \nabla\Gamma\left(X\right) = \sum_{k \in \mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma\left(X\right) \circ d\beta_k. \tag{16}$$

where Γ is assumed to be Lipschitz and smooth and such that $\Gamma\left(r\right) = 0$ on $\left(-\infty, \varepsilon\right)$ for some $\varepsilon > 0$ very mall and $\left|\Gamma\left(r\right)'\right| \leq const.$

Scaling limit of stochastic PDE

The Stratonovich noise that we introduced previously can be formulated in Itô form by the following transformation

$$\begin{aligned}\sum_{k\in\mathbb{Z}_0^2}\alpha_k\sigma_k\nabla\Gamma\left(X\right)\circ d\beta_k &= \sum_{k\in\mathbb{Z}_0^2}\alpha_k\sigma_k\nabla\Gamma\left(X\right)d\beta_k \\ &-\frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}\alpha_k^2 div\left[\left(\Gamma'\left(X\right)\right)^2\sigma_k\otimes\sigma_k\nabla X\right]dt.\end{aligned} \tag{17}$$

By applying Lemma 2.6 from

- F. Flandoli, D. Luo, *Convergence of transport noise to Ornstein–Uhlenbeck for 2D Euler equations under the enstrophy measure*. Ann. Probab. 48, no. 1, 264–295, 2020.

we have that

$$\sum_{k\in\mathbb{Z}_0^2}\alpha_k^2\left(\sigma_k\otimes\sigma_k\right)=\frac{1}{2}I_2,$$

where I_2 is a two-dimensional identity matrix.

The Stratonovich noise that we introduced previously can be formulated in Itô form by the following transformation

$$\sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X) \circ d\beta_k = \sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X)\, d\beta_k - \frac{1}{2}\sum_{k\in\mathbb{Z}_0^2} \alpha_k^2 div\left[\left(\Gamma'(X)\right)^2 \sigma_k \otimes \sigma_k \nabla X\right] dt. \quad (18)$$

We denote by

$$g(r) = \frac{1}{4}\int_0^r \left(\Gamma'(x)\right)^2 dx, \quad r \in \mathbb{R}$$

which satisfies $g(0) = 0$ and is Lipschitz from the properties of Γ.

Scaling limit of stochastic PDE

The Stratonovich noise that we introduced previously can be formulated in Itô form by the following transformation

$$\sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X) \circ d\beta_k = \sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X)\, d\beta_k - \frac{1}{2} \sum_{k\in\mathbb{Z}_0^2} \alpha_k^2 div\left[\left(\Gamma'(X)\right)^2 \sigma_k \otimes \sigma_k \nabla X\right] dt. \tag{19}$$

Going back to (19) we get that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X) \circ d\beta_k \\ = &\sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X)\, d\beta_k - \frac{1}{4} div\left[\left(\Gamma'(X)\right)^2 \nabla X\right] dt \\ = &\sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X)\, d\beta_k - \Delta g(X). \end{aligned} \tag{20}$$

Scaling limit of stochastic PDE

We can rigorously write equation as

$$\begin{cases} dX - \Delta\Psi\left(X\right)dt - \Delta g\left(X\right)dt + \sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma\left(X\right)d\beta_k = F \\ X\left(0,\xi\right) = x_0 \end{cases} \qquad (21)$$

Note that this is possible only when we work on a torus.

Definition

Let $x \in L^2\left(\Pi^2\right)$. We say that equation (21) has a weak solution if there exist

- a filtered reference probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{t\geq 0}, \mathbb{P}\right)$,
- a sequence of independent $\mathcal{F}_t$ Brownian motions,
- an $H^{-1}\left(\Pi^2\right)$ $-$valued continuous $\mathcal{F}_t-$adapted process $X \in L^2\left(0, T; L^2\left(\Pi^2\right)\right)$

and the following holds true

$$\begin{aligned}\left(X\left(t\right), e_j\right)_2 &= \left(x, e_j\right)_2 + \int_0^t \left(F\left(s\right), e_j\right)_2 ds \\ &+ \int_0^t \left(\Psi\left(X\left(s\right)\right), \Delta e_j\right)_2 ds + \int_0^t \left(g\left(X\left(s\right)\right), \Delta e_j\right)_2 ds \\ &+ \sum_{k\in\mathbb{Z}_0^2} \int_0^t \alpha_k \left(\sigma_k \Gamma\left(X\left(s\right)\right), \nabla e_j\right) d\beta_k\left(s\right).\end{aligned}$$

Scaling limit of stochastic PDE

We consider a sequence $\left\{\alpha^N\right\}_{N\in\mathbb{N}} \subseteq l^2\left(\mathbb{Z}_0^2\right)$ constructed as before and satisfying

$$\lim_{N\to\infty} \left\|\alpha^N\right\|_{l^\infty} = 0, \tag{22}$$

and we denote X^N the corresponding solutions of equation (21) with $\left\{\alpha_k^N\right\}$ instead of $\left\{\alpha_k\right\}$ in the construction of the noise.

Scaling limit of stochastic PDE

In the present work we prove that the law of X^N converges in the usual weak (or, more precisely weak-*) sense to a Dirac mass concentrated on the unique solution to the following deterministic porous media equation

$$\begin{cases} dX - \Delta\Psi(X)\,dt - \Delta g(X)\,dt = F \\ X(0,\xi) = x. \end{cases} \tag{23}$$

We can write the solution to equation (23) as a PDE weak one

$$X \in L^2\left((0,T)\times\Pi^2\right) \cap C\left([0,T]\times H^{-1}\left(\Pi^2\right)\right)$$

in the following form

$$\begin{aligned} \left(X(t), e_j\right)_2 &= \left(x, e_j\right)_2 + \int_0^t \left(F(s), e_j\right)_2 ds \\ &\quad + \int_0^t \left(\Psi(X(s)), \Delta e_j\right)_2 ds + \int_0^t \left(g(X(s)), \Delta e_j\right)_2 ds, \end{aligned}$$

for all test functions e_j provided above.

Scaling limit of stochastic PDE

Theorem

Let $\{\alpha^N\}_{N\in\mathbb{N}} \subseteq l^2\left(\mathbb{Z}_0^2\right)$ *be a sequence satisfying the assumptions above. We denote*

- *by* X^N *the corresponding solutions to the equations (21), where* α *is replaced with* (α^N), *and*
- *by* $\nu^N := \mathbb{P}_{X^N}$ *the law of these solutions supported by* $L^2\left((0,T)\times\Pi^2\right)\cap C\left([0,T];H^{-1}\left(\Pi^2\right)\right)$.

Then, the family $\{\nu^N\}$ *is tight on* $C\left([0,T];H^{-1}\left(\Pi^2\right)\right)$, *and it converges weakly to the Dirac measure* δ_X *where* X *is the unique solution of the equation (23).*

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Thank you very much !