Wasserstein distance in terms of the comonotonicity Copula

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This is a joint work with Mariem Abdellatif, Peter Kuchling, and Barbara Rüdiger.

Definition

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Theorem (Sklar (1959))
Let F be a joint distribution function with margins $F_1, \ldots, F_d$. Then there exists a copula $C : .[0,1]^d \to [0,1]$ s.t. $\forall x_i \in [-\infty, \infty]$

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \quad (1)$$

If F_i are continuous then C is unique. Otherwise, it is uniquely determined on $ranF_1 \times \cdots \times ranF_d$, where $ranF_i$ denotes the range of F_i. Conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then F defined in (1) is a joint distribution with margins $F_1, \ldots, F_d$.

Theorem
For every copula $C(u_1, \ldots, u_d)$ we have the bounds

$$\max\left\{\sum_{i=1}^{d} u_i - d + 1,\ 0\right\} \leq C(u_1, \ldots, u_d) \leq \min\{u_1, \ldots, u_d\}.$$

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Example

- *Fundamental:*
  - *Independence copula* $\Pi(u_1, \ldots, u_d) = \prod_{i=1,\ldots,d} u_i$.
  - *Comonotonicity copula* $M(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}$
  - *Countermonotonicity copula* $W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$
- *Implicit: extracted from well-known joint distributions using Sklar's Theorem.*
- *Explicit: simple closed-form expressions.*

Definition
Let μ, ν be two probability measures on a Polish space $(E, \mathcal{E}, d)$, then the **Wasserstein distance** of order $p \in [1, \infty)$ is defined as,

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{E \times E} d(x, y)^p \pi(dx, dy) \right)^{\frac{1}{p}}.$$

where $\Pi(\mu, \nu)$ is the set of all couplings between μ and ν.

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Moreover, the **p-Wasserstein space** is defined as,

$$P_p(E) := \{\mu \in \mathcal{P}(E) : \int_E d(x, x_o)^p \mu(dx) < \infty\}$$

where $x_o \in E$ is arbitrary and $\mathcal{P}(E)$ is the set of all probability measures on E.

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Theorem (Villani (2002))
Let $p \in [1, \infty)$. Let $(E, \mathcal{E}, d)$ be a Polish space. Let $\mu, \nu \in \mathcal{P}(E)$. Then, there exists an optimal coupling of (μ, ν) for $W_p(\mu, \nu)$.

Theorem
Let μ, ν be two probability measures in $P_p(\mathbb{R})$. Let F and G be the associated distribution functions. Then for all $p \in [1, +\infty)$,

$$W_p(\mu, \nu) = \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x - y|^p dM(x, y) \right)^{\frac{1}{p}}$$

where $M(x, y) = \min\{F(x), G(y)\}$.

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Theorem
Let μ, ν be two probability measures in $P_p(\mathbb{R}^d)$ sharing the same copula. Denote by F_i and G_i $(i = 1, \ldots, d)$ the distribution functions of the one-dimensional margins of μ and ν, respectively. Then for all $p \in [1, \infty)$,

$$(W_p(\mu, \nu))^p = \sum_{i=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}} |x_i - y_i|^p dM(F_i(x_i), G_i(y_i))$$

where $M(x, y) = \min\{F(x), G(y)\}$.

Thank you for your attention!

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