

# Probabilistic interpretation of McKean PDEs in the non-conservative and path-dependent case

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This talk focuses on two probabilistic interpretations of a specific nonlinear parabolic PDE

(Natalini et al. 2004-)

$$\begin{aligned}\partial_t \rho(t, x) = & \Delta \rho(t, x) - \nabla \cdot \left[ b \left( \int_0^t \mathbf{K} * \rho(s, x) ds, \int_0^t \nabla \mathbf{K} * \rho(s, x) ds \right) \rho(t, x) \right] \\ & - c \left( \int_0^t \mathbf{K} * \rho(s, x) ds \right) \rho(t, x),\end{aligned}$$

where  $K : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth **regularising kernel**.

- ▶ **Non-conservative.**
- ▶ **Path-dependent** coefficients.
- ▶ Existence and uniqueness of only **weak** solutions has been established.

Application: modeling the **sulphation phenomenon**, a degradation process affecting **marble surfaces** exposed to **atmospheric pollutants**



1908



1969

Schloss Herten, Westphalia, Germany

A probabilistic interpretation based on a  
Feynman-Kac approach

We introduce a stochastic model coupled with a **non-local**  
**Feynman-Kac-type** equation

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Probabilistic representation of a class of  
non-conservative nonlinear  
Partial Differential Equations

Anthony Le Cavil, Nadia Oudjane and Francesco Russo

We introduce a stochastic model coupled with a **non-local Feynman-Kac**-type equation

$$Y_t = Y_0 + \int_0^t b \left( \int_0^s u^m(r, Y_s) dr, \int_0^s \nabla u^m(r, Y_s) dr \right) ds + \sqrt{2} W_t, \quad t \in [0, T];$$

$$Y_0 \sim \zeta_0, \quad \mathcal{L}(\zeta_0) = \mu_0, \quad \mu_0(dx) = \rho_0(x)dx;$$

$$\mathcal{L}(Y) = m;$$

$$u^m(t, y) = \int_{\mathcal{C}} K(y - X_t(\omega)) \exp \left( - \int_0^t c \left( \int_0^s u^m(r, X_s(\omega)) dr \right) ds \right) m(d\omega),$$

where  $X$  is the canonical process.

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where  $X$  is the canonical process.

We call such an SDE the **McKean-Feynman-Kac SDE**.

To relate the McKean-Feynman-Kac SDE to the PDE model, we introduce a McKean-Vlasov SDE by a general approach<sup>1</sup>

$$Y_t = Y_0 + \int_0^t b \left( \int_0^s K * \gamma_r^m(Y_s) dr, \int_0^s \nabla K * \gamma_r^m(Y_s) dr \right) ds + \sqrt{2} W_t, \quad t \in [0, T];$$

$$Y_0 \sim \zeta_0, \quad \mathcal{L}(\zeta_0) = \mu_0, \quad \mu_0(dx) = \rho_0(x)dx;$$

$$\mathcal{L}(Y) = m;$$

$$\gamma_t^m(f) = \mathbb{E}_m \left[ \varphi(X_t) \exp \left( - \int_0^t c \left( \int_0^s K * \gamma_r^m(X_s) dr \right) ds \right) \right] \quad \forall \varphi \in C_b(\mathbb{R}).$$

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<sup>1</sup>A. Le Cavil, N. Oudjane, and F. Russo (2016)

We consider the **particle system** associated with the McKean-Feynman-Kac SDE and we establish **propagation of chaos**

Let  $N \in \mathbb{N}$ . For  $i = 1, \dots, N$ ,

$$\xi_t^{i,N} = \xi_0^{i,N} + \int_0^t b \left( \int_0^s u^{\mu_N}(r, \xi_s^{i,N}) dr, \int_0^s \nabla u^{\mu_N}(r, \xi_s^{i,N}) dr \right) ds + \sqrt{2} W_t^i;$$

$$\mathcal{L}(\xi_0^{i,N}) = \mathcal{L}(Y_0); \quad \mu_N := \frac{1}{N} \sum_{i=1}^N \varepsilon_{\xi_i^{i,N}};$$

$$u^{\mu_N}(t, y) = \frac{1}{N} \sum_{j=1}^N K(y - \xi_t^{j,N}) \exp \left( - \int_0^t c \left( \int_0^s u^{\mu_N}(r, \xi_s^{i,N}) dr \right) ds \right).$$

A probabilistic interpretation based on  
killed diffusions

The second probabilistic interpretation consists of linking the PDE model to **killed McKean-Vlasov-type SDE**

Let  $X = \{X_t\}_{t \in [0, T]}$  be a  $\mathbb{R} \cup \{\Delta\}$  - valued stochastic process solution of

$$\begin{aligned} X_t &= X_0 + \int_0^t b \left( \int_0^s K * \nu_r(X_s) dr, \int_0^s \nabla K * \nu_r(X_s) dr \right) ds + \sqrt{2} W_t, && \text{for } t \in [0, \tau) \cap [0, T], \\ X_t &\in \{\Delta\}, && \text{for } t \geq \tau, \end{aligned}$$

where

$$\nu_t = \mathbb{P}(X_t \in \cdot, \tau > t)$$

and

$$X_0 \sim \zeta_0 ; \quad \mathcal{L}(\zeta_0) = \mu_0 ; \quad \mu_0(dx) = \rho_0(x)dx.$$

The second probabilistic interpretation consists of linking the PDE model to a McKean-Vlasov-type SDE **stopped** at a random time

The stopping time  $\tau = \tau(X)$  is defined as

$$\tau = \tau(X) := \inf_{t \geq 0} \left\{ \int_0^t c \left( \int_0^s K * \nu_r(X_s) dr \right) ds \geq Z \right\},$$

where  $Z \sim \text{Exp}(1)$  is independent of  $X$ .

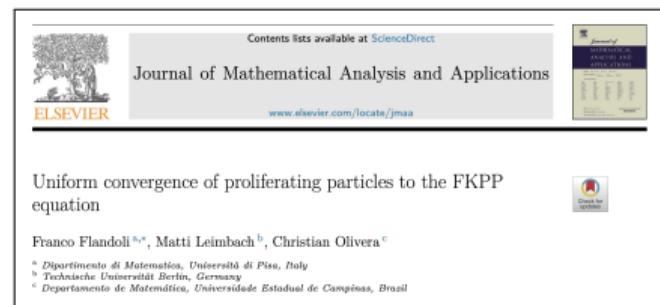
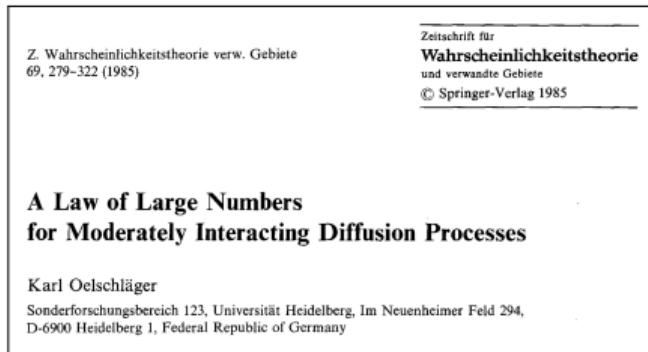
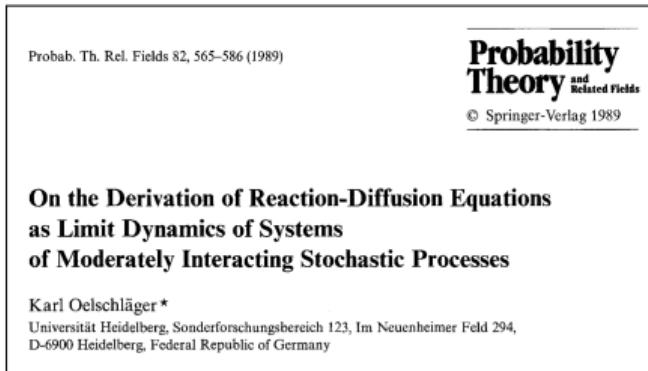
We consider the **particle system** associated with the killed McKean-Vlasov SDE and we study its **convergence**

Let  $N \in \mathbb{N}$ . For  $i = 1, \dots, N$ ,

$$\begin{aligned} \xi_t^i &= \xi_0^i + \int_0^t b \left( \int_0^s u_N(r, \xi_s^i) dr, \int_0^s \nabla u_N(r, \xi_s^i) dr \right) ds + \sqrt{2} W_t^i, && \text{for } t \in [0, \tau^i) \cap [0, T], \\ \xi_t^i &\in \{\Delta\}, && \text{for } t \geq \tau, \end{aligned}$$

- ▶  $\tau^i := \inf_{t \geq 0} \left\{ \int_0^t c \left( \int_0^s u_N(r, \xi_s^i) dr \right) ds \geq Z \right\}$ ,  $Z \sim \text{Exp}(1)$ .
- ▶  $\Gamma_t^N \subseteq \{1, \dots, N\}$  is the set of particles “alive” at time  $t$ .
- ▶  $\mu_t^N := \frac{1}{N} \sum_{i \in \Gamma_t^N} \delta_{\xi_t^i}$  is the empirical measure.
- ▶  $u_N(t, \xi_t^i) := K^N * \mu_t^N$  is the empirical density, where  $K^N(x) := N^\beta K(N^\beta x)$ .

We **rescale** the kernel with respect to  $N$  and we introduce a **parameter**  $\beta \in (0, 1/5)$ . It corresponds to a **medium range** of interaction between the particles



The particle system is **well-posed** and its empirical density **converges** to a solution to the original PDE model **without the smoothing**

### Theorem (convergence)

The empirical density of the killed particle system **converges in probability** to a weak solution of the original PDE model **without the smoothing**

$$\begin{aligned}\partial_t \rho(t, x) = & \Delta \rho(t, x) - \nabla \cdot \left[ b \left( \int_0^t \rho(s, x) ds, \int_0^t \nabla \rho(s, x) ds \right) \rho(t, x) \right] \\ & - c \left( \int_0^t \rho(s, x) ds \right) \rho(t, x),\end{aligned}$$

as the number of particles  $N \in \mathbb{N}$  grows to infinity.

If we pick  $\beta = 0$ , we are in the **mean-field** case and we have **propagation of chaos**. But the convergence is again to the **regularised model**

### Theorem (propagation of chaos)

- i)  $\mu^N \xrightarrow{L^2} \mu$ , where  $\mu$  satisfies a measure-valued version of the regularised PDE model.
- ii) For every  $k \geq 1$ ,

$$\left( \xi^{N,1}, \dots, \xi^{N,k} \right) \xrightarrow[d]{N \rightarrow \infty} (X^1, \dots, X^k),$$

where  $(X^1, \dots, X^k)$  are  $k$  copies of the limit SDE.



## Essential bibliography

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Questions?