Probabilistic interpretation of McKean PDEs in the non-conservative and path-dependent case

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Joint work with
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This talk focuses on two probabilistic interpretations of a specific nonlinear parabolic PDE

(Natalini et al. 2004-)

$$\boxed{\begin{aligned}\partial_t \rho(t,x) &= \Delta\rho(t,x) - \nabla\cdot\left[b\left(\int_0^t \mathbf{K}*\rho(s,x)ds, \int_0^t \nabla\mathbf{K}*\rho(s,x)ds\right)\rho(t,x)\right] \\ &\quad - c\left(\int_0^t \mathbf{K}*\rho(s,x)ds\right)\rho(t,x),\end{aligned}}$$

where $K : \mathbb{R} \to \mathbb{R}$ is a smooth **regularising kernel**.

- **Non-conservative**.
- **Path-dependent** coefficients.
- Existence and uniqueness of only **weak** solutions has been established.

Application: modeling the **sulphation phenomenon**, a degradation process affecting **marble surfaces** exposed to **atmospheric pollutants**

1908

1969

Schloss Herten, Westphalia, Germany

A probabilistic interpretation based on a Feynman-Kac approach

We introduce a stochastic model coupled with a **non-local Feynman-Kac**-type equation

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Probabilistic representation of a class of non-conservative nonlinear Partial Differential Equations

Anthony Le Cavil, Nadia Oudjane and Francesco Russo

We introduce a stochastic model coupled with a **non-local Feynman-Kac**-type equation

$$Y_t = Y_0 + \int_0^t b\left(\int_0^s u^m(r, Y_s)dr, \int_0^s \nabla u^m(r, Y_s)dr\right) ds + \sqrt{2}\, W_t, \qquad t \in [0, T];$$

$$Y_0 \sim \zeta_0, \quad \mathcal{L}(\zeta_0) = \mu_0, \quad \mu_0(dx) = \rho_0(x)dx;$$

$$\mathcal{L}(Y) = m;$$

$$u^m(t, y) = \int_{\mathcal{C}} K(y - X_t(\omega)) \exp\left(-\int_0^t c\left(\int_0^s u^m(r, X_s(\omega))dr\right) ds\right) m(d\omega),$$

where X is the canonical process.

We introduce a stochastic model coupled with a **non-local Feynman-Kac**-type equation

$$Y_t = Y_0 + \int_0^t b\left(\int_0^s u^m(r, Y_s)dr, \int_0^s \nabla u^m(r, Y_s)dr\right) ds + \sqrt{2}\, W_t, \qquad t \in [0, T];$$

$$Y_0 \sim \zeta_0, \quad \mathcal{L}(\zeta_0) = \mu_0, \quad \mu_0(dx) = \rho_0(x)dx;$$

$$\mathcal{L}(Y) = m;$$

$$u^m(t, y) = \int_{\mathcal{C}} K(y - X_t(\omega)) \exp\left(-\int_0^t c\left(\int_0^s u^m(r, X_s(\omega))dr\right) ds\right) m(d\omega),$$

where X is the canonical process.

We call such an SDE the **McKean-Feynman-Kac SDE**.

To relate the McKean-Feynman-Kac SDE to the PDE model, we introduce a McKean-Vlasov SDE by a general approach[1]

$$Y_t = Y_0 + \int_0^t b\left(\int_0^s K * \gamma_r^m(Y_s)dr, \int_0^s \nabla K * \gamma_r^m(Y_s)dr\right) ds + \sqrt{2}W_t, \quad t \in [0,T];$$

$$Y_0 \sim \zeta_0, \quad \mathcal{L}(\zeta_0) = \mu_0, \quad \mu_0(dx) = \rho_0(x)dx;$$

$$\mathcal{L}(Y) = m;$$

$$\gamma_t^m(f) = \mathbb{E}_m\left[\varphi(X_t)\exp\left(-\int_0^t c\left(\int_0^s K * \gamma_r^m(X_s)dr\right) ds\right)\right] \quad \forall \varphi \in C_b(\mathbb{R}).$$

[1] A. Le Cavil, N. Oudjiane, and F. Russo (2016)

We consider the **particle system** associated with the McKean-Feynman-Kac SDE and we establish **propagation of chaos**

Let $N \in \mathbb{N}$. For $i = 1, \ldots, N$,

$$\xi_t^{i,N} = \xi_0^{i,N} + \int_0^t b\left(\int_0^s u^{\mu_N}(r, \xi_s^{i,N})dr, \int_0^s \nabla u^{\mu_N}(r, \xi_s^{i,N})dr\right) ds + \sqrt{2}\, W_t^i\,;$$

$$\mathcal{L}\left(\xi_0^{i,N}\right) = \mathcal{L}\left(Y_0\right); \qquad \mu_N := \frac{1}{N}\sum_{i=1}^{N} \varepsilon_{\xi_\cdot^{i,N}};$$

$$u^{\mu_N}(t,y) = \frac{1}{N}\sum_{j=1}^{N} K\left(y - \xi_t^{j,N}\right) \exp\left(-\int_0^t c\left(\int_0^s u^{\mu_N}(r, \xi_s^{i,N})dr\right) ds\right).$$

A probabilistic interpretation based on killed diffusions

The second probabilistic interpretation consists of linking the PDE model to **killed McKean-Vlasov-type SDE**

Let $X = \{X_t\}_{t\in[0,T]}$ be a $\mathbb{R} \cup \{\Delta\}$ - valued stochastic process solution of

$$\begin{cases} X_t = X_0 + \int_0^t b\left(\int_0^s K * \nu_r(X_s)\, dr,\ \int_0^s \nabla K * \nu_r(X_s)\, dr\right) ds + \sqrt{2}W_t, & \text{for } t \in [0,\tau) \cap [0,T], \\ X_t \in \{\Delta\}, & \text{for } t \geq \tau, \end{cases}$$

where

$$\nu_t = \mathbb{P}\left(X_t \in \cdot, \tau > t\right)$$

and

$$X_0 \sim \zeta_0\,; \quad \mathcal{L}(\zeta_0) = \mu_0\,; \quad \mu_0(dx) = \rho_0(x)dx.$$

The second probabilistic interpretation consists of linking the PDE model to a McKean-Vlasov-type SDE **stopped** at a random time

The stopping time $\tau = \tau(X)$ is defined as

$$\tau = \tau(X) := \inf_{t \geq 0} \left\{ \int_0^t c\left(\int_0^s K * \nu_r(X_s) dr \right) ds \geq Z \right\},$$

where $Z \sim Exp(1)$ is independent of X.

We consider the **particle system** associated with the killed McKean-Vlasov SDE and we study its **convergence**

Let $N \in \mathbb{N}$. For $i = 1, \dots, N$,

$$
\begin{cases}
\xi_t^i = \xi_0^i + \int_0^t b\left(\int_0^s u_N\left(r, \xi_s^i\right) dr, \int_0^s \nabla u_N\left(r, \xi_s^i\right) dr\right) ds + \sqrt{2} W_t^i, & \text{for } t \in [0, \tau^i) \cap [0, T], \\
\xi_t^i \in \{\Delta\}, & \text{for } t \geq \tau,
\end{cases}
$$

- $\tau^i := \inf_{t \geq 0} \left\{ \int_0^t c\left(\int_0^s u_N(r, \xi_s^i) dr\right) ds \geq Z \right\}$, $Z \sim Exp(1)$.
- $\Gamma_t^N \subseteq \{1, \dots, N\}$ is the set of particles "alive" at time t.
- $\mu_t^N := \frac{1}{N} \sum_{i \in \Gamma_t^N} \varepsilon_{\xi_t^i}$ is the empirical measure.
- $u_N\left(t, \xi_t^i\right) := K^N * \mu_t^N$ is the empirical density, where $\boxed{K^N(x) := N^\beta K(N^\beta x)}$.

We **rescale** the kernel with respect to N and we introduce a **parameter** $\beta \in (0, 1/5)$. It corresponds to a **medium range** of interaction between the particles

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Probability Theory and Related Fields

On the Derivation of Reaction-Diffusion Equations as Limit Dynamics of Systems of Moderately Interacting Stochastic Processes

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Z. Wahrscheinlichkeitstheorie verw. Gebiete 69, 279–322 (1985)

Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete

A Law of Large Numbers for Moderately Interacting Diffusion Processes

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Uniform convergence of proliferating particles to the FKPP equation

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The particle system is **well-posed** and its empirical density **converges** to a solution to the original PDE model **without the smoothing**

Theorem (convergence)

The empirical density of the killed particle system **converges in probability** to a weak solution of the original PDE model **without the smoothing**

$$\begin{aligned}\partial_t \rho(t,x) \ &= \Delta\rho(t,x) - \nabla \cdot \left[b\left(\int_0^t \rho(s,x)ds, \int_0^t \nabla\rho(s,x)ds \right) \rho(t,x) \right] \\ &\quad - c\left(\int_0^t \rho(s,x)ds \right) \rho(t,x),\end{aligned}$$

as the number of particles $N \in \mathbb{N}$ grows to infinity.

If we pick $\beta = 0$, we are in the **mean-field** case and we have **propagation of chaos**. But the convergence is again to the **regularised** model

Theorem (propagation of chaos)

i) $\mu^N \xrightarrow{L^2} \mu$, where μ satisfies a measure-valued version of the regularised PDE model.

ii) For every $k \geq 1$,

$$\left(\xi^{N,1}, \dots, \xi^{N,k}\right) \xrightarrow[d]{N\to\infty} (X^1, \dots, X^k),$$

where $(X^1, \dots, X^k)$ are k copies of the limit SDE.

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Questions?