The Boltzmann Process

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dedicated to Errico Presutti with gratitude. Joint results based on

- [ARS23] Albeverio, S., R. B., Sundar P.: On the construction and identifcation of Boltzmann processes, *Ensaios Mathemáticos*, 38, 1-22, Papers in honor of Errico Presutti, Sociedade Brasileira de Mathematica (2023).
- [RS24] R. B., Sundar P.: Identification and existence of Boltzmann processes, submitted, arxiv 2301.08662v2

The Boltzmann Eq. was introduced by L. Boltzmann (1844 -1906) to describe the dynamics of the density in position and velocity of particles of a diluted gas expanding in vacuum. Boltzmann assumed that any gas molecule (particle) travels straight until an elastic collision occurs with another particle. In Boltzmann model only binary centered collisions (in the same point x) are taken into account.

Aim: Identify the stochastic process which describes the dynamics in position and velocity of **one tagged particle**. I.e indentify the stochastic process whose density function solves the Boltzmann eq., or distribution satisfies the weak Boltzmann eq., i.e identify the "Boltzmann process".

ARS23 : identifies the Boltzmann process as solution of an SDE of McKean Vlasov type

RS24 : proves existence of Boltzmann processes

Let f be the particle density function, which depends on time $t \geq 0$, the **space variable** $x \in \mathbb{R}^3$, and the **velocity variable** $z \in \mathbb{R}^3$ of the point particle. $\{f(t,x,z)\}_{t\in[0,T]}$ (or $t \in \mathbb{R}_+^0$) solves the Boltzmann equation (BE) if

$$\frac{\partial f}{\partial t}(t,x,z) + z \cdot \nabla_x f(t,x,z) = Q(f,f)(t,x,z) \quad \forall t \in [0,T]$$

where

$$Q(f,f)(t,x,z)$$
$$= \int_{\mathbb{R}^3\times S^2} \{f(t,x,z^\star)f(t,x,v^\star) - f(t,x,z)f(t,x,v)\} B(|z-v|,n)dndv$$

- $z, v =$ pre-collision velocities.
- $z^\star$, $v^\star =$ post-collision velocities.

- $\mathbf{n} = \frac{v^\star - v}{|v^\star - v|}$ deflection of velocity of incoming particle;
- Conservation of momentum and energy hold (where here the Mass $m = 1$):

$$\begin{cases} z^\star + v^\star = z + v \\ (z^\star)^2 + (v^\star)^2 = z^2 + v^2 \end{cases}$$

- $$\begin{cases} v^\star = v + (\mathbf{n}, z - v)\mathbf{n} \\ z^\star = z - (\mathbf{n}, z - v)\mathbf{n} \end{cases}$$

- Jump size $= \alpha(z, v, \mathbf{n}) = (\mathbf{n}, \mathbf{z} - \mathbf{v})\mathbf{n}$
- The scattering measure $B(|z - v|, n) dndv$ gives the collision rate. In Boltzmann model $B(|z - v|, n) dn = |(\mathbf{n}, z - v)| d\mathbf{n}$

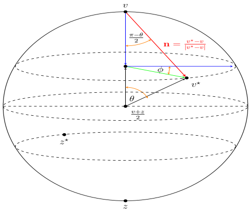


Figure 1: Parameterization of collisions

1

$\mathbf{n}$ depends on θ, ϕ (e.g. spherical coordinates with $\theta =$ colatitude angle; $\phi =$ longitude angle). Center of the ball is at $\frac{z+v}{2} = \frac{z^\star + v^\star}{2}$. $(\mathbf{n}, z - v) = |z - v| cos(\frac{\pi}{2} - \frac{\theta}{2}) = |z - v| sin(\frac{\theta}{2})$ with $\mathbf{n} := \frac{v^\star - v}{|v^\star - v|}$.

Boltzmann hard sphere model: (recall $\mathbf{n} = \frac{v^\star - v}{|v^\star - v|}$)

$$(\mathbf{n}, z - v) = |z - v| sin(\frac{\theta}{2})$$

hence

$$B(|z - v|, n) dndv := (\mathbf{n}, z - v) d\mathbf{n} dv$$

corresponds in polar coordinates to

$$B(z, dv, d\theta) d\phi := |z - v| dv \, sin(\frac{\theta}{2}) cos(\frac{\theta}{2}) d\theta d\phi$$

$$(\theta, \phi) \in \Xi := (0, \pi] \times [0, 2\pi)$$

This is the "cut -off hard sphere model" by Boltzmann.
When identifying the Boltzmann process, we consider the more general hard sphere (cut -off and non cut -off) model:

$$B(z, dv, d\theta) d\phi := |z - v| dv \; Q(d\theta) d\phi \quad \text{with} \int_0^\pi \theta Q(\theta) < \infty$$

$$\frac{\partial f}{\partial t}(t,x,z) + u \cdot \nabla_x f(t,x,z) = Q(f,f)(t,x,z) \qquad (1)$$

$$Q(f,f) = \int_\Lambda \{f(t,x,z^\star)f(t,x,v^\star) - f(t,x,z)f(t,x,v)\}B(|v-z|,n)dvdn.$$

$$\Lambda := \mathbb{R}^3 \times (0,\pi] \times [0,2\pi)$$

$$\begin{cases} v^\star = v + \alpha(u,v,\theta,\phi) \\ z^\star = z - \alpha(z,v,\theta,\phi) \end{cases}$$

with $\theta \in [0,\pi]$ longitute ; $\phi \in [0,2\pi)$ colatitude angles,

$$\alpha(z,v,\theta,\phi) := (n, z-v)n, \quad n = \frac{v^\star - v}{|v^\star - v|}$$

$$B(|v-z|,n)dvdn = |z-v|dvQ(d\theta)d\phi$$

The Problem: Stochastic interpretation of Boltzmann eq.

ARS23 Identify stochastic processes $\{(Z_t, R_t)\}$ whose density function solves the Boltzmann equation,

RS24 and prove its existence.

Stochastic interpretation of spatially homogeneous Boltzmann eq.

$$\frac{\partial h(t,z)}{\partial t} = \int_{\mathbb{R}^3} \int_{S^2} \left(h(t,v^\star) h(t,z^\star) - h(t,v) h(t,z) \right) B(|z-v|, n) dn dv.$$

H. Tanaka '73, '78, '87; Funaki '85 and ...A. Bressan, C. Cercignani, N. Fournier, J. Horowitz, L. Karandikar, R. Esposito, R. Marra, S. Meleard, Y. Morimoto, C.Mouhot, M. Pulvirenti, F. Rezakhanlou, C. Villani, S. Wang, T. Yang....

Stochastic interpretation of Boltzmann -Enskog eq.: [S. Albeverio, B. R., P. Sundar JSP 2017] (identification of Boltzmann - Enskog process and $\exists$ for Maxwellian case); [M. Friesen, B. R., P. Sundar NoDEA 2019] ($\exists$ of Boltzmann -Enskog process in general)

$$\frac{\partial f}{\partial t}(t,x,z) + u \cdot \nabla_x f(t,x,z) = Q(f,f)(t,x,z) \qquad (2)$$

$$Q(f,f) = \int_\Lambda \{f(t,x,z^\star)f(t,x,v^\star) - f(t,x,z)f(t,x,v)\}B(|v-z|,n)dvdn.$$

$$\Lambda := \mathbb{R}^3 \times (0,\pi] \times [0,2\pi)$$

Illner, Shinbrot *Comm.Math. Phys* 1984; Illner , Pulvirenti *Comm.Math. Phys* 1989 :
Assume $0 \leq f(0,x,z) \leq b \exp\left(-\beta_0(x^2+z^2)\right)$ with $b > 0$, $\beta_0 > 0$,
then $\exists$ of solution globally in time for Boltzmann equation in cut-off hard sphere model and
$0 \leq f(t,x,z) \leq (C \cdot b) \exp\left(-\beta_0(x-tz)^2\right)$ a.e.

Weak formulation of the Boltzmann - equation

$$\int_{\mathbb{R}^6} \psi(x,z)\frac{\partial f}{\partial t}(t,x,z)dxdz - \int_{\mathbb{R}^6} f(t,x,z)(z,\nabla_x\psi(x,z))dxdz$$
$$= \int_{\mathbb{R}^6} f(t,x,z)L_f\psi(x,z)dxdz \qquad (3)$$

for all $\psi \in C_0^2(\mathbb{R}^6)$ and for all $t \in [0,T]$ with

$$L_f\psi(x,z) = \int_{\mathbb{R}^3\times\Xi} \{\psi(x,z^\star) - \psi(x,z)\} f(t,x,v)B(z,dv,d\theta)d\phi$$

with

$$B(z,dv,d\theta)d\phi = |z-v|dvQ(d\theta)d\phi$$

Proposition

[Tanaka '82] Let $\Psi(x,z) \in C_0(\mathbb{R}^6)$,

$$\int_{\mathbb{R}^9\times\Xi} \Psi(x,z) f(x,z^\star,t) f(x,v^\star,t) B(z,dv,d\theta)\, dx dz d\phi$$
$$=$$
$$\int_{\mathbb{R}^9\times\Xi} \Psi(x,z^\star) f(x,z,t) f(x,v,t) B(z,dv,d\theta)\, dx dz d\phi$$

where $\Xi := [0,\pi] \times (0,2\pi]$

Weak formulation of the Boltzmann - equation

Let $\mu_t(dx, dz) := f(t, x, z)dxdz$,

$$\int_{\mathbb{R}^6} \psi(x, u)\mu_t(dx, dz) - \int_0^t \int_{\mathbb{R}^6} (z, \nabla_x \psi(x, z))\mu_s(dx, dz)ds$$
$$= \int_0^t \int_{\mathbb{R}^6} L_f \psi(x, z)\mu_s(dx, dz)ds \qquad (4)$$

for all $\psi \in C_0^2(\mathbb{R}^6)$ and for all $t \in [0, T]$ with

$$L_f \psi(x, z) = \int_{\mathbb{R}^3 \times \Xi} \{\psi(x, z^\star) - \psi(x, z)\} f(t, x, v) B(z, dv, d\theta) d\phi$$

with

$$B(z, dv, d\theta)d\phi = |z - v| dv Q(d\theta) d\phi$$

This suggests the infinitesimal generator of a Boltzmann process is given by $(z, \nabla_x) + L_f$.

Definition

A collection of densities $\{f(t,x,z\}_{t\in[0,T]}$ (resp. distributions $\mu_t(x,z\}_{t\in[0,T]})$, with $x,z\in\mathbb{R}^3$, is a strong (resp. weak) solution of the Boltzmann equation in $[0,T]$, if for any $t\in[0,T]$ it solves (2) (resp. (4)).

We denote by $\mathbb{D} := \mathbb{D}([0,T],\mathbb{R}^3)$ the space of all right continuous functions with left limits on $[0,T]$ taking values in $\mathbb{R}^3$, and equipped with the topology induced by the Skorohod metric.

Definition

A stochastic process $(X_s,Z_s)_{s\in[0,T]}$ with values on $\mathbb{D}\times\mathbb{D}$, having density $\{f(t,x,z)\}_{t\in[0,T]}$ (resp. distribution $\mu_t(x,z\}_{t\in[0,T]})$ which solves the Boltzmann Equation (2) (or (4))) is called a "Boltzmann process".

We remark that the infinitesimal generator of a Boltzmann process is given by $(z,\nabla_x)+L_f$. **This suggests that the Boltzmann process solves a McKean -Vlasov -type** SDE.

Preparing ARS23

Let $U_0 = \mathbb{D} \times [0, \pi) \times (0, 2\pi]$..
The physical model of Boltzmann suggests that the Boltzmann process $(X_., Z_.)$ should be a sol. of the McKean -Vlasov SDE

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0} \alpha(Z_s, v, \theta, \phi) d\mathbf{N}, \end{cases} \tag{5}$$

where in the above equation, $d\mathbf{N} := \mathbf{N}(\mathbf{dv}, \mathbf{d}\theta, \mathbf{d}\phi, \mathbf{dr}, \mathbf{ds})$ is a Poisson random measure with **random** compensator $f(s, X_s, v)|Z_s - v| dv Q(d\theta) d\phi ds$ and with $(X_t, Z_t) \sim f(t, x, z) dx dz$.

This is however equivalent to say that the Boltzmann process $(X_., Z_.)$ is a martingale sol. of McKean Vlasov SDE given in terms of a Noise term with Poisson measure with deterministic compensator, defined through $m(t,v)$ which denotes the marginal density of velocity v at time t (i.e. $m(t,v) := \int_{\mathbb{R}^3} f(t,x,v)dx$, so that upon desintagration of measures $f(t,x|v)m(t,v)$ $:= f(t,x,v)$)

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0 \times \mathbb{R}_+^0} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\, |Z_s - v_s| f(s, X_s|v_s)]}(r) dN, \end{cases} \tag{6}$$

where in the above equation, $dN := N(dv, d\theta, d\phi, dr, ds)$ is a Poisson random measure with compensator $m(s,v)dvQ(d\theta)d\phi dsdr$.

Theorem

[ARS23] *Let* $\{f(t,x,v)\}_{t\in\mathbb{R}^0_+}$ *be a collection of densities which satisfies hypothesis* **B**. *Let* X_0 *and* Z_0 *be* $\mathbb{R}^3$*- valued random variables. Suppose that for* $T > 0$ *there exists a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$*, an adapted process* $(X_t, Z_t)_{t\in[0,T]}$ *with values on* $\mathbb{D}\times\mathbb{D}$*, with* $(X_t, Z_t) \sim f(t,x,z)dxdz$*, and which satisfies in* $[0,T]$ *the SDE*

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0\times\mathbb{R}^0_+} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\ |Z_s - v_s| f(s, X_s|v_s)]}(r) dN, \end{cases} \tag{7}$$

where in the above equation, $dN := N(dv, d\theta, d\phi, dr, ds)$ *is a Poisson random measure with compensator* $m(s,v)dvQ(d\theta)d\phi dsdr$*. Then* $(X_t, Z_t)_{t\in[0,T]}$ *is a Boltzmann process.*

Assumptions

Let $\{f(t,x,v)\}_{t\in\mathbb{R}_+^0}$ be a collection of densities on $(\mathbb{R}^3\times\mathbb{R}^3,\mathcal{B}(\mathbb{R}^3\times\mathbb{R}^3))$.

Hypotheses B:

B$_1$. $t \to f(t,x,v)$ is differentiable for each $x,v\in\mathbb{R}^3$ fixed;

B$_2$. $f(t,x,v)$ is jointly continuous in (t,x,v) and $\forall s_0\in\mathbb{R}_+^0$ $f(s,\cdot,\cdot)\to f(s_0,\cdot,\cdot)$ in $L^1(\mathbb{R}^6)$, as $s\to s_0$;

B$_3$. $\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|u|^2)f(s,x,u)du\in C([0,T])\quad\forall\ T>0.$

Proof: We can apply the Itô formula to $(X_s, Z_s)_{s\in\mathbb{R}_+}$. In fact let t, $\Delta t > 0$, $\psi \in C_0^2(\mathbb{R}^3 \times \mathbb{R}^3)$, then

$$\psi(X_{t+\Delta t}, Z_{t+\Delta t})$$
$$= \psi(X_t, Z_t) + \int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s))ds$$
$$+ \int_t^{t+\Delta t} \int_{U_0\times\mathbb{R}_0^+} \{\psi(X_s, Z_s + \alpha(Z_s, v_s, \theta, \xi) 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s|v_s)]}(r))$$
$$- \psi(X_s, Z_s)\} dN$$

$$\mathbb{E}[\psi(X_{t+\Delta t}, Z_{t+\Delta t}) - \psi(X_t, Z_t)] =$$
$$\mathbb{E}\left[\int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s))ds\right] +$$
$$\mathbb{E}\left[\int_t^{t+\Delta t}\int_{U_0} \{\psi(X_s, Z_s + \alpha(Z_s, v_s, \theta, \xi)) - \psi(X_s, Z_s)\} \sigma(|Z_s - v_s|) f(s, X_s, v_s) dv Q(c\right.$$

Upon dividing by Δt on both sides, we obtain (3)

[RS24]]: Existence of a weak sol. of (7), i.e of a Boltzmann process.

Difficulties:

- SDE is of Mc Kean -Vlasov type, because the density $f(t,x,z)$ of $\{X_t, Z_t\}_{t\in[0,T]}$ appears through the velocity marginal in the compensator of the Poisson measure N.
- No Lipschitz coefficients

$$|\alpha(z,v,n) - \alpha(\tilde{z},\tilde{v},\tilde{n}| \not\leq C(|z-\tilde{z}| + |v-\tilde{v}|)$$

- $\alpha(z,v,n)$ is not bounded; $\quad |\alpha(z,v,n)| = |z-v| sin(\frac{\theta}{2})$

bypass the problem that the SDE associated to BE is of McKean -Vlasov type

We reduce the problem in finding a solution of an SDE, which on the r.h.s equals the SDE before, but is not of McKean -Vlasov type, i.e. the distribution of its sol. $(X_\cdot, Z_\cdot)$ is not assumed to be $f(t,x,z)dxdz$, but some $g(t,x,z)dxdz$:

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0 \times \mathbb{R}_+^0} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s | v_s)]}(r) dN, \end{cases} \tag{8}$$

We prove however $g(t,x,z) = f(t,x,z)\ \forall t$ **if** $f(t,x,z)$ **solves the Boltzmann equation.**

Theorem

[RS24] Let $T > 0$ *and* $\{f(t,x,v)\}_{t\in[0,T]}$ *be a collection of densities which solves the Boltzmann equation* (2) *(and satisfies regularity hypotheses* **B***). Suppose that for fixed* $T > 0$ *there exists a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$*, an adapted process* $(X_t, Z_t)_{t\in[0,T]}$ *with values on* $\mathbb{D} \times \mathbb{D}$*, such that it satisfies a.s.* (12) *and has time marginals* $\{g(t,x,z)dxdz\}_{t\in[0,T]}$*, then* $\{g(t,x,z)\}_{t\in[0,T]}$ *solves the bilinear BE in* $[0,T]$*.*

"bilinear BE" associated to $\{f(t,x,v)\}_{t\in[0,T]}$:

$$\frac{\partial g}{\partial t}(t,x,z) + z \cdot \nabla_x g(t,x,z) = Q(f,g)(t,x,z) \tag{9}$$

where

$$Q(f,g)(t,x,z)$$
$$= \int_{\mathbb{R}^3 \times \Xi} \{g(t,x,z^\star)f(t,x,v^\star) - g(t,x,z)f(t,x,z)\} B(|z-v|, n) dndv$$

Consider the following "bilinear BE" associated to $\{f(t,x,v)\}_{t\in[0,T]}$

$$\frac{\partial g}{\partial t}(t,x,z) + z\cdot\nabla_x g(t,x,z) = Q(f,g)(t,x,z) \qquad (10)$$

where

$$Q(f,g)(t,x,z) = \int_{\mathbb{R}^3\times\Xi}\{g(t,x,z^\star)f(t,x,v^\star) - g(t,x,z)f(t,x,z)\}B(|z-v|,n)dndv$$

Theorem (RS24)

Assume $\{f(t,x,z)\}_{[0,T]}$ solves the Boltzman equation. Assume $\{g(t,x,z)\}_{[0,T]}$ solves (10) and $g(0,x,z)=f(0,x,z)$. Assume g,f satisfy conditons **C1**. *Then $f(t,x,z)= g(t,x,z)$ a.s. $\forall t\in[0,T]$.*

Assumption **C1.**

- $\{g(t,x,u)\}_{[0,T]}$ satisfies like $\{f(t,x,v)\}_{[0,T]}$ conditions **B**
- The densities $f(t,x,z)$ and $g(t,x,z)$ are in $C^{1,2}([0,T] \times \mathbb{R}^6)$ and are strictly positive-valued functions a.e.
- $g \log g, g \log f \in L^1(\mathbb{R}^6)$ for each $t \in [0,T]$ and $\lim_{|x| \to \infty} g(t,x,z) = 0$

proof:

prove

$$R_t(g|f) := \int_{\mathbb{R}^6} \log\left(\frac{g(t.x.z)}{f(t,x,z)}\right) g(t,x,z)dxdz = 0 \quad \forall t \in [0,T] \tag{11}$$

This implies $f(t,x,v) = g(t,x,v)$ λ -a.s. $\forall t \in [0,T]$

Recall:

Recall that for any two probability measures μ, ν on a common measurable space $(X, \mathcal{X})$, the relative entropy of ν with respect to μ, denoted $R(\nu \,||\, \mu)$, is defined by

$$R(\nu \,||\, \mu) = \int_X \left(\log \frac{d\nu}{d\mu} \right) d\nu$$

if ν is absolutely continuous with respect to μ. Otherwise, we set $R(\nu \,||\, \mu) = \infty$. The following Lemma is well known.

Lemma

Let μ, ν be two probability measures on a measurable space $(X, \mathcal{X})$. Then $R(\nu \,||\, \mu) \geq 0$ and $R(\nu \,||\, \mu) = 0$ if and only if $\mu = \nu$.

Existence of solution

The problem is reduced to find a solution of

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0 \times \mathbb{R}_+^0} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\, \sigma(|Z_s - v_s|) f(s, X_s | v_s)]}(r) dN, \end{cases} \tag{12}$$

where in the above equation, $dN := N(dv, d\theta, d\phi, dr, ds)$ is a Poisson random measure with compensator $m(s,v)dvQ(d\theta)d\phi dsdr$, with $m(t,v) := \int_{\mathbb{R}^3} f(t,x,v)dx$ and $\{f(t,x,z)\}_{t\in[0,T]}$ **solves the Boltzmann equation.**

Definition

A "weak solution" of equation (12) in the time interval $[0, T]$ is a triplet $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P}), N(dv, d\theta, d\phi, dr, ds), (X_t, Z_t)_{t\in[0,T]})$ for which the following properties hold:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$ is a stochastic basis;
- $N(dv, d\theta, d\phi, dr, ds)$ is an adapted Poisson random measure with compensator $m(s, v)dvQ(d\theta)d\phi dsdr$;
- $(X_\cdot, Z_\cdot) := (X_t, Z_t)_{t\in[0,T]})$ is an adapted càdlàg stochastic process with valued in $\mathbb{R}^d \times \mathbb{R}^d$ which satisfies (12)P -a.s.

Let $\{f(t,x,z)\}_{t\in[0,T]}$ be any collection of densities which satisfies the following conditions:

B4.

$$\sup_{s\in[0,T],x\in\mathbb{R}^3}\int_{\mathbb{R}^3} f(s,x,v)dv \le C_T < \infty.$$

B5. For every $T>0$ and $K>0$, there exists a constant $C_T^K>0$ such that

$$\sup_{s\in[0,T],|x|\le K}\int_{\mathbb{R}^3} max(1,|v|^2)|\nabla_x f(s,x,v)|dv \le C_T^K < \infty.$$

B6.

$$\sup_{s\in[0,T],x\in\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^3 f(s,x,v)dv \le c_T < \infty.$$

Theorem (RS24)

Let $T > 0$ and $\{f(t,x,v)\}_{t\in[0,T]}$ be a collection of densities which satisfy $f(t,x,v) \in C([0,T]\times\mathbb{R}^6)$ and Hypotheses **B4** *-***B6***. Let the initial distribution of (X_0, Z_0) admit finite second moment. There exists a weak solution of*

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0\times\mathbb{R}^0_+} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s|v_s)]}(r) dN, \end{cases}$$

where in the above equation, $dN := N(dv, d\theta, d\phi, dr, ds)$ is a Poisson random measure with compensator $m(s,v)dvQ(d\theta)d\phi dsdr$, with $m(t,v) := \int_{\mathbb{R}^3} f(t,x,v)dx$. Moreover,

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t|^2] + \sup_{t\in[0,T]} \mathbb{E}[|Z_t|^2] < \infty$$

Proof was done in several steps.

Let $j \in \mathbb{N}$, $B_j := \{z \in \mathbb{R}^3 : |z| \leq j\}$ and

$$\alpha_j(z, v, \theta, \phi) := \alpha(\frac{z}{1 + d(z, B_j)}, v, \theta, \phi) \tag{13}$$

$$\sigma_j(z, v) := \sigma(|\frac{z}{1 + d(z, B_j)} - v|) \tag{14}$$

where $d(z, B_j)$ denotes the distance of $z \in \mathbb{R}^3$ from B_j. Solve first (by Picard iteration and by involving Tanakas Parameter transformation)

$$\mathcal{X}_t^j = \mathcal{X}_0 + \int_0^t \mathcal{Z}_s^j ds \quad \forall t \in [0, T]. \tag{15}$$

$$\mathcal{Z}_t^j = \mathcal{Z}_0 + \int_0^t \int_U \alpha_j(\mathcal{Z}_{s\_}^j, v, \theta, \phi) \times 1_{[0,\, \sigma_j(\mathcal{Z}_{s\_}^j, v) f(s, \mathcal{X}_s^j|v)]}(r) N(dv, d\theta, d\phi, dr, ds) \quad \forall t \in [0, T] \tag{16}$$

Remark

$$\frac{|z|}{1 + d(z, B_j)} \leq \min(j, |z|) \tag{17}$$

and there exists a constant $K_j > 0$, such that

$$\left| \frac{z}{1 + d(z, B_j)} - \frac{z'}{1 + d(z', B_j)} \right| \leq K_j |z - z'|. \tag{18}$$

Theorem

[RS24] Let $f(t,x,u) \in C([0,T] \times \mathbb{R}^6)$, and $\{f(t,x,v)\}_{t\in[0,T]}$ be a collection of densities which satisfies hypotheses **B4** - **B6**. *Let $j \in \mathbb{N}$ be fixed.*
For any stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ which supports an adapted Poisson noise $\mathcal{N}(dv, d\theta, d\phi, dr, ds)$ with compensator $m(s,v)dvQ(d\theta)d\phi dsdr$ and any random vector $(\mathcal{X}_0, \mathcal{Z}_0)$ on $(\Omega, \mathcal{F}_0, P)$ with values on $\mathbb{R}^3 \times \mathbb{R}^3$ having finite second moment, there exists on the same filtered space an adapted Poisson random measure $N^j(dv, d\theta, d\phi, dr, ds)$ with compensator $m(s,v)dv$ $Q(d\theta)d\phi drds$ and a stochastic process $(\mathcal{Z}_., \mathcal{X}_.)$ which solves a.s.

$$\mathcal{X}_t^j = \mathcal{X}_0 + \int_0^t \mathcal{Z}_s^j ds \quad \forall t \in [0,T]. \tag{19}$$

$$\mathcal{Z}_t^j = \mathcal{Z}_0 + \int_0^t \int_U \alpha_j(\mathcal{Z}_{s\_}^j, v, \theta, \phi) \times 1_{[0,\, \sigma_j(\mathcal{Z}_{s\_}, v) f(s, \mathcal{X}_s | v)]}(r) N^j(dv, d\theta, d\phi, dr, ds) \quad \forall t \in [0,T] \tag{20}$$

Proof of Theorem 9:

Introduces a Picard Iteration which uses a parameter transformation of Tanaka to bypass the problem that the noise coefficients are not Lipschitz.

$$|\alpha(z, v, n) - \alpha(\tilde{z}, \tilde{v}, \tilde{n}| \nleq C(|z - \tilde{z}| + |v - \tilde{v}|)$$

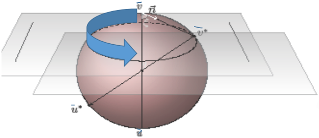
bypass the problem of no-Lipschitz coefficients

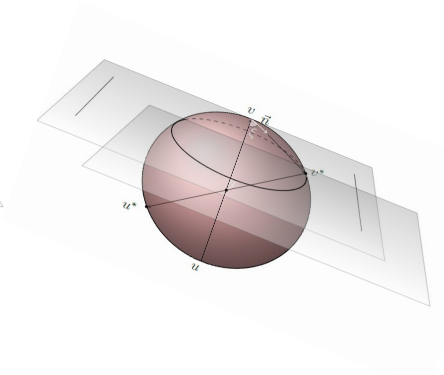
No Lipschitz coefficients

$$|\alpha(z,v,n) - \alpha(\tilde{z},\tilde{v},\tilde{n}| \not\leq C(|z-\tilde{z}| + |v-\tilde{v}|)$$

H. Tanaka proved that under a certain parameterization, we have for all $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}^d$, and all $\theta \in [0,\pi]$ and $\phi \in (0, 2\pi]$,

$$|\alpha(z,v,\theta,\phi) - \alpha(\tilde{z},\tilde{v},\theta,\phi + \phi_0(z-v,\tilde{z}-\tilde{v},\phi))| \leq 2\theta(|v-\tilde{v}| + |z-\tilde{z}|).$$





Tanaka remarked that a weak sol. $(X_t, Z_t)_{t\in[0,T])}$ of

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0\times\mathbb{R}_+^0} \alpha(Z_s, v_s, \theta, \phi+\phi_0)1_{[0,\ \sigma(|Z_s-v_s|)f(s,X_s|v_s)]}(r)dN, \end{cases} \tag{21}$$

with ϕ_0 being the Tanaka parametrization is also a weak sol. of

$$\begin{cases} X_t = X_0 + \int_0^t Z_s ds \\ Z_t = Z_0 + \int_0^t \int_{U_0\times\mathbb{R}_+^0} \alpha(Z_s, v_s, \theta, \phi)1_{[0,\ \sigma(|Z_s-v_s|)f(s,X_s|v_s)]}(r)dN^0, \end{cases} \tag{22}$$

where in both cases the Noises are Poisson random measure with compensator $m(s, v)dvQ(d\theta)d\phi dsdr$.

Remove the truncation using finite kinetic energy

First prove

Theorem (RS24)

Let $(\mathcal{X}^j_\cdot, \mathcal{Z}^j_\cdot)$ be a solution of ((19), (20)). There exist constants $K_T \geq 0$, $M_T \geq 0$, $k_T > 0$ and $m_T > 0$ **which do not depend on** *j and such that the following inequalities holds*

$$\sup_{t\in[0,T]} \mathbb{E}[|\mathcal{Z}^j_t|^2] \leq (M_T + \mathbb{E}[|\mathcal{Z}_0|^2])e^{TK_T} \tag{23}$$

$$\mathbb{E}[\sup_{t\in[0,T]} |\mathcal{Z}^j_t|] \leq Tk_T(M_T + \mathbb{E}[|\mathcal{Z}_0|^2])e^{TK_T} + m_T + \mathbb{E}[|\mathcal{Z}_0|]. \tag{24}$$

Proof.

Use Ito formula to compute $\mathbb{E}[|\mathcal{Z}^j_t|^2]$ and symmetry properties. □

Let

$$\tau_1 = \inf\{t \in [0, T] : |\mathcal{Z}^1_{t_-}| > 1\}.$$

We denote the solution $(\mathcal{X}^1_\cdot, \mathcal{Z}^1_\cdot)$ as $(X^{(1)}, Z^{(1)})$ in the time interval $[0, \tau_1)$. Its law is a probability measure P_1 on the canonical path space $(\Omega, \mathcal{F}_{\tau_1})$, $\Omega = \mathbb{D} \times \mathbb{D}$. Next, for each fixed $\omega \in \Omega$, consider the equations (20), (19) with initial condition given by $(X^1_{\tau_1(\omega)}, Z^1_{\tau_1(\omega)})$ with initial time being $\tau_1(\omega)$. The above procedure can be used to obtain a solution $(X^{(2,\omega)}, Z^{(2,\omega)})$ until time $\tau_{2,\omega}$ where

$$\tau_{2,\omega} = \inf\{t \in [\tau_1(\omega), T] : |Z^{(2,\omega)}_{t_-}| > 2\}.$$

The law of $(X^{(2,\omega)}_\cdot, Z^{(2,\omega)}_\cdot)$ until $\tau_{2,\omega}$ is denoted by Q_ω, a probability measure for each $\omega \in \Omega$. It gives us a regular conditional probability distribution (rcpd) given $\mathcal{F}_{\tau_1}$. Hence, by a theorem of Stroock and Varadhan, we can patch the measures P_1 and Q so that there exists a unique probability measure P_2 on $(\Omega, \mathcal{F}_{\tau_2})$ such that $P_2 = P_1$ on $\mathcal{F}_{\tau_1}$, and the rcpd of P_2 given $\mathcal{F}_{\tau_1}$ is Q.

Use

$$\mathbb{E}[\sup_{t\in[0,T]} |\mathcal{Z}_t^j|] \leq Tk_T(M_T + \mathbb{E}[|\mathcal{Z}_0|^2])e^{TK_T} + m_T + \mathbb{E}[|\mathcal{Z}_0|].$$

to prove

$$\mathrm{P}(\cup_{j\in\mathbb{N}}\{\tau_j = T\}) = 1. \tag{25}$$

Since $\mathrm{P}(\tau_j \leq \tau_{j+1}) = 1 \ \forall j \in \mathbb{N}$, it follows $\lim_{n\to\infty} \tau_n = T$ a.s. and hence the proof is over.

Proof of (25)**:**

$$\begin{aligned}\mathrm{P}(\tau_j < T) &= \mathrm{P}(\sup_{t\in[0,T]} |Z_t^j| > j) \\ &\leq \frac{\mathbb{E}[\sup_{t\in[0,T]} |Z_t^j|]}{j} \\ &\leq \frac{Tk_T(M_T + \mathbb{E}[|\mathcal{Z}_0|^2])e^{TK_T} + m_T + \mathbb{E}[|\mathcal{Z}_0|]}{j}\end{aligned}$$

where we have used (24). It follows

$$\mathrm{P}(\cap_{j\in\mathbb{N}}\{\tau_j < T\}) = \lim_{j\to\infty} \mathrm{P}(\{\tau_j < T)\} = 0. \tag{26}$$

THANK YOU!

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