L^p Convergence of a Numerical Scheme for SDEs with Distributional Coefficients

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SDEs with Distributional Drift

Given $b \in C_T \mathcal{C}^{(-\beta)+}\left(\mathbb{R}^d\right)$, $\beta \in (0, 1/2)$ the expression

$$X_t = X_0 + W_t + \text{“}\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{b}(\mathbf{t}, \mathbf{X_t})\mathbf{dt}\text{”} \quad X_0 \sim \mu, \tag{1}$$

can be given meaning in many ways. In [Issoglio and Russo 2024b] this is done using a Zvonkin-type transformation. They prove that the martingale problem with distributional drift for (1) is equivalent to:

$$Y_t = Y_0 + \lambda \int_0^t Y_s ds - \int_0^t \psi(s, Y_s) ds + \int_0^t \nabla\phi(s, \psi(s, Y_s)) dW_s, \tag{2}$$

where ϕ and ψ, are both **functions** and are the solution to a Kolmogorov-type PDE and its space-inverse, respectively.

A Two-Step Numerical Scheme for Distributional SDEs

From now on, fix $d = 1$, this ensures that the solutions to (1) are **strong** solutions. Also, consider $b \in C_T^{1/2}\mathcal{C}^{(-\beta)+}(\mathbb{R})$.

- The first step is to regularise b using the **heat semigroup**. Define

$$b^N := P_{\frac{1}{N}} b.$$

  Notice that b^N is a function, so we obtain a sequence of strong solutions X^N of SDEs:

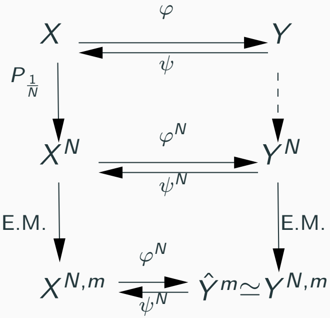
$$X_t^N = X_0 + W_t + \int_0^t b^N(t, X_t^N)dt, \quad N \geq 1.$$

- The second step consists of applying the classical Euler-Maruyama scheme to the regularised SDEs:

$$X_{t_{i+1}}^{N,m} = X_{t_i}^{N,m} + (t_{i+1} - t_i)b^N(t_i, X_{t_i}^{N,m}) + (W_{t_{i+1}} - W_{t_i}).$$

Sketch of the Proof for the Convergence Rate

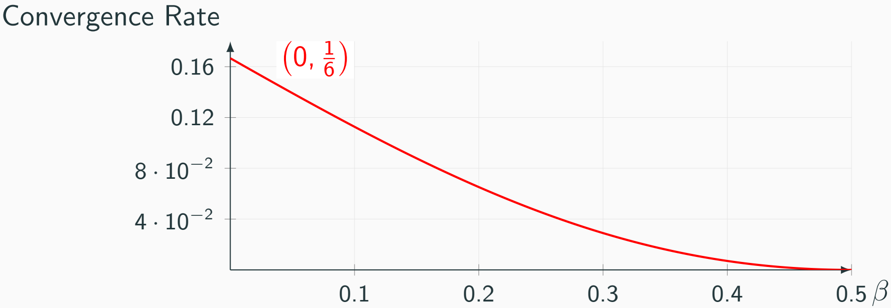
Since b is a distribution, it is difficult to obtain quantitative estimates of the speed of convergence of the approximation $X^{N,m}$ to X. To solve this, it is useful to pass to the associated process Y.
During the calculations, estimates of **local times** are necessary, highlighting how this technique is specific to $d = 1$.



L^1 Convergence Rate of the Scheme

In [Jáquez, Issoglio, and Palczewski 2023] N, is set to m^γ and optimised for maximum overall rate of convergence. They obtain:

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{N(m),m} - X_t\right|\right] \leq m^{-\frac{\left(\frac{1}{2}-\beta-\varepsilon\right)^2}{1+\beta+\varepsilon+2\left(\frac{1}{2}-\beta-\varepsilon\right)^2}} \quad (3)$$

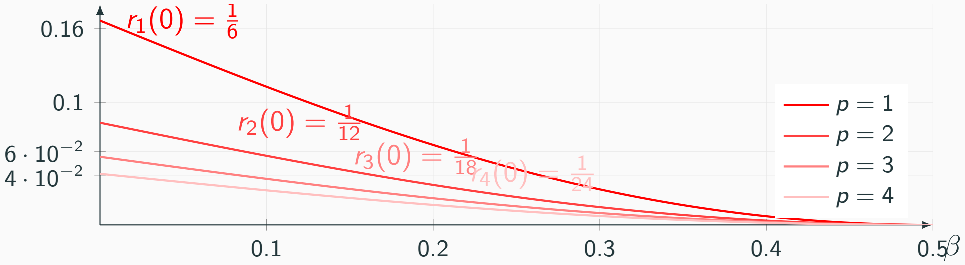


L^p Convergence of the Numerical Scheme

The result from before allows to treat the case where $p \neq 1$. Modifying the results in [Jáquez, Issoglio, and Palczewski 2023] allows to obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{N(m),m} - X_t\right|^p\right]^{\frac{1}{p}} \le c m^{-\frac{\left(\frac{1}{2}-\beta-\varepsilon\right)^2}{p\left(\varepsilon+\beta+1+2\left(\frac{1}{2}-\beta-\varepsilon\right)^2\right)}} \tag{4}$$

Convergence Rate



A Gronwall-type Lemma

Lemma (Gyöngy and Rásonyi 2011, Lemma 3.2)

Let $(Z_t)_{t\geq 0}$ be a non-negative stochastic process and set $V(t) = \sup_{s\leq t} Z_s$. Assume that for some $p > 0$, $q \geq 1$, $\rho \in [1, q]$, and constants K and $\delta \geq 0$

$$\mathbb{E}\left[V_t^p\right] \leq K\mathbb{E}\left[\left(\int_0^t V_s ds\right)^p\right] + K\mathbb{E}\left[\left(\int_0^t Z_s^\rho ds\right)^{p/q}\right] + \delta < +\infty,$$

for all $t \geq 0$. Then, for each $T \geq 0$ the following holds:
If $p \geq q$ or both $\rho < q$ and $p > q + 1 - \rho$ hold then there exist constants C_1 and C_2 depending on K, T, ρ, and p such that

$$\mathbb{E}\left[V_T^p\right] \leq C_1\delta + C_2 \int_0^T \mathbb{E}\left[Z_s\right] ds. \tag{5}$$

Conclusions and Future Perspectives

- We have shown that the results of L^1 convergence of [Jáquez, Issoglio, and Palczewski 2023] can be extended to the case of $L^p - \sup$ convergence, although the rate deteriorates as p increases.
- One possible application of this result is to prove L^p convergence of numerical schemes for Mckean-Vlasov SDEs with distributional coefficients.
- There are recent results that prove similar bounds and find constant rates of convergence for SDEs that have a distributional drift, but fail in the case of Brownian noise, see [Goudenège, Haress, and Richard 2024].

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Thank you for your attention!

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