Nonlinear Markov processes in the sense of McKean

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Reference:

- Rehmeier/R.: arXiv: 2212.12424v2, to appear in J. Theor. Probab.
  Barbu/Rehmeier/R.: arXiv: 2409.18744v2, in revision at AOP
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$$\frac{\partial}{\partial t} u(t,x,y) = \Delta_x u(t,x,y),\ (t,x) \in (0,\infty) \times \mathbb{R}^d,$$
$$u(0,x,y) = \delta_y(x)\ (= \text{Dirac measure in } y \in \mathbb{R}^d).$$

Solution: Classical **heat kernel**

$$u(t,x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\, e^{-\frac{1}{4t}|x-y|^2},\ (t,x) \in (0,\infty) \times \mathbb{R}^d.$$

GENERAL
Linear Parabolic PDE (more precisely: **linear** Fokker-Planck equation)

$\updownarrow$

P R O B A B I L I T Y

Wiener measure $\mathbb{W}_y$ on $C([0,\infty);\mathbb{R}^d)_y$ [Wiener 1923]
For $W(t) : C([0,\infty);\mathbb{R}^d)_y \to \mathbb{R}^d$,
$W(t)(w) := w(t),\ t \geq 0,$

$$\underbrace{(W(t))_*(\mathbb{W}_y)}_{\text{"push forward"}}(\mathrm{d}x) = u(t,x,y)\mathrm{d}x,\ t > 0,$$

$$(W(0))_*(\mathbb{W}_y) = \delta_y$$

$(W(t))_{t\geq 0}, \mathbb{W}_y)_{y\in\mathbb{R}^d}$ "Brownian motion"

Markov process!

linear Markov process (described by SDE)

0. Motivation and longterm programme: Nonlinear case

A N A L Y S I S

Core example: parabolic p–Laplace equation on $\mathbb{R}^d$ with $p > 2$:

$$\frac{\partial}{\partial t} u(t,x,y) = \operatorname{div}(|\nabla u|^{p-2} \nabla u)(t,x,y),\ (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

$$u(0,x,y) = \delta_y(x) \ (= \text{Dirac measure in } y \in \mathbb{R}^d).$$

Solution: **Barenblatt solution**

$$u(t,x,y) = t^{-k} \Big(C_1 - q t^{-\frac{kp}{d(p-1)}} |x-y|^{\frac{p}{p-1}} \Big)_+^{\frac{p-1}{p-2}},$$

$(t,x) \in (0,\infty) \times \mathbb{R}^d$, where $k := \left(p - 2 + \frac{p}{d}\right)^{-1}$,

$q := \frac{p-2}{p} \left(\frac{k}{d}\right)^{\frac{1}{p-1}}$ and $C_1 > 0$ s.th. $\int_{\mathbb{R}^d} u(t,x,y)\, dx = 1$.

P R O B A B I L I T Y

Our result: $\exists$ prob. measure P_y on $C([0,\infty); \mathbb{R}^d)_y$ s. th.

$$\underbrace{(X(t))_*(P_y)}_{\text{“push forward”}}(dx) = u(t,x,y)\, dx,\ t > 0,\ \text{(McKean!)}$$

where $X = \big(X(t)\big)_{t \geq 0}$ is the solution of

$$\mathbf{d}\boldsymbol{X}(\boldsymbol{t}) = \boldsymbol{\nabla}(|\boldsymbol{\nabla u}(\boldsymbol{t}, \boldsymbol{X}(\boldsymbol{t}), \boldsymbol{y})|^{\boldsymbol{p-2}})\mathbf{d}\boldsymbol{t}$$

$$+ |\boldsymbol{\nabla u}(\boldsymbol{t}, \boldsymbol{X}(\boldsymbol{t}), \boldsymbol{y})|^{\frac{p-2}{2}} \mathbf{d}\boldsymbol{W}(\boldsymbol{t}),\ t > 0,\ (X(0))_*(P_y) = \delta_y.$$

$((X(t))_{t \geq 0}, P_y)_{y \in \mathbb{R}^d}$ “p–Brownian motion”

Nonlinear Markov process!

GENERAL

Nonlinear Parabolic PDE (more precisely: **nonlinear** Fokker-Planck equation)

$\updownarrow$

nonlinear (time-inhomogeneous) Markov process (described by MVSDE)

1. Introduction: Definition of a nonlinear Markov process

Define for $s \geq 0$

$$\Omega_s := C([s,\infty),\mathbb{R}^d) = \text{ space of continuous paths in } \mathbb{R}^d \text{ starting at time } s$$

with Borel $\sigma-$algebra $\mathcal{B}(\Omega_s)$ and for $\tau \geq s$

$$\pi^s_\tau : \Omega_s \to \mathbb{R}^d, \quad \pi^s_\tau(w) := w(\tau),\ w \in \Omega_s$$

and for $r \geq s$

$$\mathcal{F}_{s,r} := \sigma(\pi^s_\tau |\ s \leq \tau \leq r).$$

Definition ([McKean: PNAS 1966])

Let $\mathcal{P}_0 \subseteq \mathcal{P}(\mathbb{R}^d)$. A nonlinear Markov process is a family $(\mathbb{P}_{(s,\zeta)})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ of probability measures $\mathbb{P}_{(s,\zeta)}$ on $\mathcal{B}(\Omega_s)$ such that

(i) *The marginals $\mathbb{P}_{(s,\zeta)} \circ (\pi^s_t)^{-1} =: \mu^{s,\zeta}_t$ belong to $\mathcal{P}_0$ for all $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{P}_0$.*

(ii) *The nonlinear Markov property holds, i.e., for all $0 \leq s \leq r \leq t$, $\zeta \in \mathcal{P}_0$*

$$\mathbb{P}_{(s,\zeta)}(\pi^s_t \in A|\mathcal{F}_{s,r})(\cdot) = p_{(s,\zeta),(r,\pi^s_r(\cdot))}(\pi^r_t \in A) \quad \mathbb{P}_{(s,\zeta)} - \text{a.s. for all } A \in \mathcal{B}(\mathbb{R}^d), \qquad \text{(MP)}$$

where $p_{(s,\zeta),(r,y)}, y \in \mathbb{R}^d$, is a regular conditional probability kernel from $\mathbb{R}^d$ to $\mathcal{B}(\Omega_r)$ of $\mathbb{P}_{(r,\mu^{s,\zeta}_r)}[\,\cdot\,|\pi^r_r = y]$, $y \in \mathbb{R}^d$ (i.e., in particular $p_{(s,\zeta),(r,y)} \in \mathcal{P}(\Omega_r)$ and $p_{(s,\zeta),(r,y)}(\pi^r_r = y) = 1$).

The term *nonlinear* Markov property originates from the fact that in the situation of the above definition the map $\mathcal{P}_0 \ni \zeta \mapsto \mu^{s,\zeta}_t$ is, in general, not convex.

Remark 3

(i) *The one-dimensional time marginals* $\mu_t^{s,\zeta} = \mathbb{P}_{(s,\zeta)} \circ (\pi_t^s)^{-1}$ *of a nonlinear Markov process satisfy the* **flow property**, *i.e.*

$$\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall\, 0 \leq s \leq r \leq t, \zeta \in \mathcal{P}_0.$$

(ii) *In the linear case the above definition coincides with the classical definition of a (linear) Markov process and the above flow property corresponds to the classical Chapman–Kolmogorov equations.*

2. Nonlinear Fokker–Planck equations (FPEs) and McKean–Vlasov stochastic differential equations (MVSDEs)

Let $\mathcal{P}(\mathbb{R}^d)$ denote the space of all Borel probability measures on $\mathbb{R}^d$, and for $1 \leq i,j \leq d$ consider measurable maps

$$b_i, a_{ij} : [0,\infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$$

such that the matrix $(a_{ij})_{i,j}$ is pointwise symmetric and nonnegative definite. Then, for $(s,\zeta) \in [0,\infty) \times \mathcal{P}(\mathbb{R}^d)$ a **nonlinear FPE** is an equation of type

$$\frac{\partial}{\partial t}\mu_t^{s,\zeta} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(a_{ij}(t,x,\mu_t^{s,\zeta})\mu_t^{s,\zeta}) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i(t,x,\mu_t^{s,\zeta})\mu_t^{s,\zeta}),\ (t,x) \in [s,\infty)\times\mathbb{R}^d, \tag{FPE}$$

where the solution $[s,\infty) \ni t \mapsto \mu_t^{s,\zeta}$ is a weakly continuous curve in $\mathcal{P}(\mathbb{R}^d)$ with some specified initial condition $\mu_0 = \zeta$.
(FPE) is meant in the weak sense of Schwartz distributions. More precisely:

Definition (see [Bogachev/Krylov/R./Shaposhnikov: AMS-Monograph 2015] and the references therein)

(i) *A* **distributional solution** *to* (FPE) *with starting time* $s \in [0,\infty)$ *and initial condition* ζ *is a weakly continuous curve* $(\mu_t^{s,\zeta})_{t\geq s}$ *of signed Borel measures on* $\mathbb{R}^d$ *of bounded variation such that* $(t,x) \mapsto a_{ij}(t,x,\mu_t^{s,\zeta})$ *and* $(t,x) \mapsto b_i(t,x,\mu_t^{s,\zeta})$ *are measurable on* $(s,\infty)\times\mathbb{R}^d$*,*

$$\int_s^t \int_{\mathbb{R}^d} \sum_{i,j=1}^d \left(|a_{ij}(r,x,\mu_r^{s,\zeta})| + |\sum_{i=1}^d b_i(r,x,\mu_r^{s,\zeta})| \right) \mu_r^{s,\zeta}(dx)dr < \infty, \ \forall t \geq s,$$

and $\forall t \geq s$

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t^{s,\zeta} = \int_{\mathbb{R}^d} \varphi \, d\zeta$$
$$+ \int_s^t \int_{\mathbb{R}^d} \left(\sum_{i,j=1}^d a_{ij}(r,x,\mu_r^{s,\zeta}) \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi(x) + \sum_{i=1}^d b_i(r,x,\mu_r^{s,\zeta}) \frac{\partial}{\partial x_i}\varphi(x) \right) \mu_r^{s,\zeta}(dx)dr,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$*. It is called* **probability solution***, if, in addition,* ζ *and each* $\mu_t^{s,\zeta}$*,* $t \geq s$*, are in* $\mathcal{P}(\mathbb{R}^d)$*.*

(ii) *Suppose* $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{R}^d)$ *such that for each* $(s,\zeta) \in [0,\infty)\times\mathcal{P}_0$ *there exists a probability solution* $[s,\infty) \ni t \mapsto \mu_t^{s,\zeta} \in \mathcal{P}_0$ *with initial condition* ζ *such that the* **flow property**

$$\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall\, 0 \leq s \leq r \leq t,\ \zeta \in \mathcal{P}_0$$

holds. Then $(\mu^{s,\zeta})_{(s,\zeta)\in[0,\infty)\times\mathcal{P}_0}$ *is called* **a solution flow of** *(FPE) in* $\boldsymbol{\mathcal{P}_0}$*.*

The (in space) dual operator to the operator on the right hand side of (FPE) is called the corresponding Kolmogorov operator L_μ for $\mu \in \mathcal{P}(\mathbb{R}^d)$, i.e. its action on test functions $\varphi \in C_0^\infty(\mathbb{R}^d)$ is given as

$$L_\mu \varphi(t,x) = \sum_{i,j=1}^{d} a_{ij}(t,x,\mu) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \varphi(x) + \sum_{i=1}^{d} b_i(t,x,\mu) \frac{\partial}{\partial x_i} \varphi(x), \tag{K}$$

where $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

In turn, this operator determines the corresponding McKean–Vlasov SDE (see [Carmona/Delarue: Springer Vol. I and II 2018] and the references therein)

$$dX^{s,\zeta}(t) = b(t, X^{s,\zeta}(t), \mu_t^{s,\zeta})dt + \sigma(t, X^{s,\zeta}(t), \mu_t^{s,\zeta})dW(t), \; t > s, \tag{MVSDEa}$$

$$\mathcal{L}_{X^{s,\zeta}(t)} = \mu_t^{s,\zeta}, \; t \geq s, \tag{MVSDEb}$$

where $\sigma = (\sigma_{ij})_{ij}$ with $\sigma\sigma^\top = (a_{ij})_{ij}$, $b = (b_1, ..., b_d)$, $W(t)$, $t \geq s$, is a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the maps $X^{s,\zeta}(t) : \Omega \to \mathbb{R}^d$, $t \geq s$, form the continuous in t solution process to (MVSDEa) such that its one-dimensional time marginals

$$\boldsymbol{\mathcal{L}_{X^{s,\zeta}(t)} := (X^{s,\zeta}(t))_* \mathbb{P}, \; t \geq 0,}$$

i.e. the push forward or image measures of $\mathbb{P}$ under $X^{s,\zeta}(t)$, satisfy (MVSDEb).

Correspondence: McKean–Vlasov SDE $\longleftrightarrow$ nonlinear FPE

a) McKean–Vlasov SDE $\longrightarrow$ nonlinear FPE:

Consider (MVSDEa,b) and **assume there exists a weak solution $\boldsymbol{X^{s,\zeta}}$**. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Then by Itô's formula, since $\mu_t^{s,\zeta} = (X^{s,\zeta}(t))_*\mathbb{P}$, $t \geq s$,

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\mu_t^{s,\zeta}(\mathrm{d}x) &= \int_\Omega \varphi(X^{s,\zeta}(t)(\omega))\,\mathbb{P}(\mathrm{d}\omega)\\ &\underset{\text{Itô}}{=} \int_\Omega \varphi(X^{s,\zeta}(0)(\omega))\,\mathbb{P}(\mathrm{d}\omega) + \int_\Omega \int_s^t L_{\mathcal{L}_{X^{s,\zeta}(r)}}\varphi(X^{s,\zeta}(r)(\omega))\,\mathrm{d}r\,\mathbb{P}(\mathrm{d}\omega)\\ &= \int_{\mathbb{R}^d} \varphi(x)\zeta(\mathrm{d}x) + \int_s^t \int_{\mathbb{R}^d} L_{\mu_r^{s,\zeta}}\varphi(x)\mu_r^{s,\zeta}(\mathrm{d}x)\mathrm{d}r\end{aligned}$$

Hence $(\mu_t^{s,\zeta})_{t\geq 0}$ is a **distributional solution** of (FPE), more precisely a **probability solution**.

b) Nonlinear FPE $\longrightarrow$ McKean–Vlasov SDE:

Theorem 0 ([Barbu/R: SIAM 2018, AOP 2020])

Let $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d)$ *and* **assume there exists a probability solution** $[s, \infty) \ni t \mapsto \mu_t^{s,\zeta} \in \mathcal{P}(\mathbb{R}^d)$ *of* (FPE). *Then there exists a d-dimensional* $(\mathcal{F}_t)$*-Brownian motion* $W(t)$, $t \geq s$, *on a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq s}, \mathbb{P})$ *and a continuous* $(\mathcal{F}_t)$*-progressively measurable map* $X^{s,\zeta} : [s, \infty) \times \Omega \to \mathbb{R}^d$ *satisfying (MVSDEa,b).*

Proof.

Nonlinear version of [Trevisan: EJP 2016] (generalizing [Figalli: JFA 2008]). □

Remark

b, σ assumed to be **only measurable** in measure variable !

3. Main result: A general condition for path laws of MVSDE–solutions to form an nonlinear Markov process

Key: Look at the **linearized equation** corresponding to (FPE), i.e. for any weakly continuous curve $[s,\infty) \ni t \mapsto \eta_t \in \mathcal{P}(\mathbb{R}^d)$ consider

$$\frac{\partial}{\partial t}\nu_t^{s,\zeta} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(a_{ij}(t,x,\eta_t)\,\nu_t^{s,\zeta}\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(b_i(t,x,\eta_t)\,\nu_t^{s,\zeta}\right), \quad (t,x)\in[s,\infty)\times\mathbb{R}^d \tag{ℓ_ηFPE}$$

where the solution $[s,\infty) \ni t \mapsto \nu_t^{s,\zeta} \in \mathcal{P}(\mathbb{R}^d)$ is a weakly continuous curve with initial condition $\zeta \in \mathcal{P}(\mathbb{R}^d)$. (Again, ($\ell_\eta$FPE) is meant in the Schwartz distribution sense!)

Now for $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{R}^d)$ let $(\mu^{s,\zeta})_{(s,\zeta)\in[0,\infty)\times\mathcal{P}_0}$ be a **solution flow** of (FPE) in $\mathcal{P}_0$ and choose specifically $\eta := \mu^{s,\zeta}$ with $(s,\zeta) \in [0,\infty)\times\mathcal{P}_0$ and consider

$$\frac{\partial}{\partial t}\nu_t^{s,\zeta} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(a_{ij}(t,x,\mu_t^{s,\zeta})\,\nu_t^{s,\zeta}\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(b_i(t,x,\mu_t^{s,\zeta})\,\nu_t^{s,\zeta}\right), (t,x)\in[s,\infty)\times\mathbb{R}^d,$$

$$\nu_s^{s,\zeta} = \zeta.$$

Then clearly, $\nu^{s,\zeta} := \mu^{s,\zeta}$ is a solution to $(\ell_{\mu^{s,\zeta}} FPE)$.

For $(s,\zeta) \in [0,\infty) \times \mathcal{P}_0$ define

$$M^{s,\zeta}_{\mu^{s,\zeta}} := \text{set of all probability solutions to } \ell_{\mu^{s,\zeta}} FPE$$

Then $M^{s,\zeta}_{\mu^{s,\zeta}}$ is a convex set. Define

$$M^{s,\zeta}_{\mu^{s,\zeta},ex} := \text{set of all extreme points of } M^{s,\zeta}_{\mu^{s,\zeta}}$$

Now we can formulate our main result.

Theorem I ([Rehmeier/R.: JTP 2025+])

Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{R}^d)$. Assume:

($\mathcal{P}_0$−Flow/lin$_{ex}$) *There exists a solution flow $(\mu^{s,\zeta})_{(s,\zeta)\in[0,\infty)\times\mathcal{P}_0}$ of (FPE) in $\mathcal{P}_0$ such that*

$$\mu^{s,\zeta} \in M^{s,\zeta}_{\mu^{s,\zeta},ex} \text{ for every } (s,\zeta) \in [0,\infty) \times \mathcal{P}_0.$$

Then:

(i) *For every $(s,\zeta) \in [0,\infty) \times \mathcal{P}_0$ the corresponding (MVSDEa,b) has a unique weak solution $X^{s,\zeta}$ with time marginal laws*

$$\mathcal{L}_{X^{s,\zeta}(t)} = \mu_t^{s,\zeta}, \quad t \geq s.$$

(ii) *The path laws*

$$\mathbb{P}_{(s,\zeta)} := \mathcal{L}_{X^{s,\zeta}} := (X^{s,\zeta})_*(\mathbb{P}), \quad (s,\zeta) \in [0,\infty) \times \mathcal{P}_0$$

form a nonlinear Markov process.

Corollary I

Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{R}^d)$ and $(\mu^{s,\zeta})_{(s,\zeta)\in[0,\infty)\times\mathcal{P}_0}$ be **a solution flow of (FPE) in $\boldsymbol{\mathcal{P}_0}$**. *Let $\tilde{\mathcal{P}}_0 \subset \mathcal{P}_0$ and assume:*

($\boldsymbol{\mathcal{P}_0}$−Flow/lin$_{\text{ex}}$) $(\mu^{s,\zeta})_{(s,\zeta)\in[0,\infty)\times\tilde{\mathcal{P}}_0}$ *is a solution flow of (FPE) in $\tilde{\boldsymbol{\mathcal{P}}}_0$ such that*

$$\mu^{s,\zeta} \in M^{s,\zeta}_{\mu^{s,\zeta},ex} \text{ for all } (s,\zeta) \in [0,\infty) \times \tilde{\boldsymbol{\mathcal{P}}}_0$$

and

($\tilde{\boldsymbol{\mathcal{P}}}_0$ − smoothing) *For every $(s,\zeta) \in [0,\infty) \times \boldsymbol{\mathcal{P}_0}$*

$$\mu^{s,\zeta}_t \in \tilde{\mathcal{P}}_0 \quad \forall\, t > s.$$

Then for every $(s,\zeta) \in [0,\infty) \times \boldsymbol{\mathcal{P}_0}$ the corresponding (MVSDEa,b) has a weak solution $X^{s,\zeta}$ with time marginal laws

$$\mathcal{L}_{X^{s,\zeta}}(t) = \mu^{s,\zeta}_t, \quad t \geq s, \qquad (*)$$

such that the path laws

$$\mathbb{P}_{(s,\zeta)} = \mathcal{L}_{X^{s,\zeta}}, \quad (s,\zeta) \in [0,\infty) \times \mathcal{P}_0,$$

form a nonlinear Markov process. Moreover, for $(s,\zeta) \in [0,\infty) \times \tilde{\boldsymbol{\mathcal{P}}}_0$, the above weak solution $X^{s,\zeta}$ is unique in law among all weak solutions satisfying $(*)$.

4. Example
4.1 FPE = parabolic p-Laplace equation

Key Step: Identify the parabolic p–Laplace equation as a nonlinear FPE for $p > 2$.

Recall: Coefficients in FPE only need to be measurable in μ. So, if for the solutions μ_t, $t \geq 0$, we have $\mu_t(dx) = u(t,x)dx$, $t > 0$, we can allow dependencies as

$$\begin{aligned} a_{ij}(t,x,\mu_t) &= \tilde{a}_{ij}(t,x,\Gamma_1(u)(t,x)), \\ b_i(t,x,\mu_t) &= \tilde{b}_i(t,x,\Gamma_2(u)(t,x)), \end{aligned} \tag{$**$}$$

where $\tilde{b}_i, \tilde{a}_{ij} : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ are measurable and each Γ_i is a functional on the space of distributional solutions whose values are again measurable functions of t and x. Noting that

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(\nabla(|\nabla u|^{p-2}u) - \nabla(|\nabla u|^{p-2})u),$$

we can rewrite the parabolic p-Laplace equation (see "first motivation page") as

$$\frac{\partial}{\partial t} u(t,x) = \Delta(|\nabla u(t,x)|^{p-2}u(t,x)) - \operatorname{div}(\nabla(|\nabla u(t,x)|^{p-2})u(t,x)),\ (t,x) \in (0,\infty)\times\mathbb{R}^d. \tag{p-LE}$$

Hence we see that (p-LE) is of type (FPE) with a_{ij}, b_i as in $(**)$, where

$$\begin{aligned} \tilde{a}_{ij}(t,x,\Gamma_1(u)(t,x)) &= \delta_{ij}|\nabla u(t,x)|^{p-2}, \\ \tilde{b}(t,x,\Gamma_2(u)(t,x)) &= \nabla(|\nabla u(t,x)|^{p-2}). \end{aligned}$$

Apply Corollary I

To the solution flow $(u^{s,\zeta})_{(s,\zeta)\in[0,\infty)\times\mathcal{P}_0}$ *of (p-LE) (= special FPE) given by the famous* **Barenblatt solution** *(see [Kamin/Vázquez 1988])*

$$u^{s,\delta_y}(t,x) := (t-s)^{-k}\left(C_1 - q(t-s)^{-\frac{kp}{d(p-1)}}|x-y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}},\ (t,x)\in(s,\infty)\times\mathbb{R}^d,$$

where $k := \left(p-2+\frac{p}{d}\right)^{-1}$, $q := \frac{p-2}{p}\left(\frac{k}{d}\right)^{\frac{1}{p-1}}$, $C_1\in(0,\infty)$ *such that* $\int_{\mathbb{R}^d} u^{s,\delta_y}(t,x)dx = 1$ *for all* $t>0$, *and* $f_+ := \max(f,0)$. *Here*

$$\mathcal{P}_0 := \tilde{\mathcal{P}}_0 \cup \{\delta_y : y\in\mathbb{R}^d\}$$

and

$$\tilde{\mathcal{P}}_0 := \{u^{0,\delta_y}(\epsilon,x)dx \,|\, \epsilon\in(0,\infty), y\in\mathbb{R}^d\}.$$

Then we obtain that for every $(s,\zeta)\in[0,\infty)\times\mathcal{P}_0$ *the corresponding (MVSDEab) (see "second motivation page") has a weak solution* $X^{s,\zeta}$ *with time marginals*

$$\mathcal{L}_{X^{s,\zeta}(t)} = u^{s,\zeta}(t,x)dx, \quad t\geq s,$$

(which is unique in law if $(s,\zeta)\in[0,\infty)\times\boldsymbol{\tilde{\mathcal{P}}_0}$*) such that the path laws*

$$\mathbb{P}_{(s,\zeta)} = \mathcal{L}_{X^{s,\zeta}}, \quad (s,\zeta)\in[0,\infty)\times\boldsymbol{\mathcal{P}_0},$$

form a nonlinear Markov process.

Remark

In this particular case it turns out that:

(i) *The solution $X^{s,\zeta}$ is unique in law for all $(s,\zeta) \in [0,\infty) \times \boldsymbol{\mathcal{P}_0}$.*

(ii) *Due to time translation invariance*

$$\mathbb{P}_{(s,\zeta)} = \mathbb{P}_{(0,\zeta)} \circ \hat{\Pi}_s^{-1} \quad \forall\, (s,\zeta) \in [0,\infty) \times \mathcal{P}_0,$$

where

$$\hat{\Pi}_s : C([0,\infty);\mathbb{R}^d) \to C([s,\infty);\mathbb{R}^d)$$

$$\hat{\Pi}_s(w(t))_{t\geq 0}) := (w(t-s))_{t\geq s}, \quad w \in C([0,\infty);\mathbb{R}^d.$$

(iii) *For $\zeta = u^{0,\delta_y}(\epsilon,x)dx \in \tilde{\mathcal{P}}_0$*

$$\mathbb{P}_{(0,\zeta)} = \mathbb{P}_{(0,\delta_y)} \circ \Pi_\epsilon^{-1},$$

where

$$\Pi_\epsilon : C([0,\infty) \times \mathbb{R}^d) \to C([0,\infty) \times \mathbb{R}^d)$$

$$\Pi_\epsilon((w(t)_{t\geq 0}) := (w(t+\epsilon))_{t\geq 0}, \quad w \in C([s,\infty) \times \mathbb{R}^d.$$

Set

$$\mathbb{P}_y := \mathbb{P}_{(0,\delta_y)}, \quad y \in \mathbb{R}^d.$$

Then $\{\mathbb{P}_y \mid y \in \mathbb{R}^d\}$ **uniquely determines** *the nonlinear Markov process $\{\mathbb{P}_{(s,\zeta)} | (s,\zeta) \in [0,\infty) \times \mathcal{P}_0\}$. Therefore, we call $\mathbb{P}_y$, $y \in \mathbb{R}^d$, p-***Brownian motion***.*

4.2 FPE = generalized porous media equation

[Barbu/R.: JFA 2021 and 2023]

Nonlinear Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t} u^{s,\zeta}(t,x) - \Delta_x(\beta(u^{s,\zeta}(t,x))) + \mathrm{div}_x(D(x)b(u^{s,\zeta}(t,x))u^{s,\zeta}(t,x)) = 0,$$
$$\forall\,(t,x) \in (s,\infty)\times\mathbb{R}^d. \qquad \text{(FPE)}$$
$$u^{s,\zeta}(s,x)dx := \zeta \in \mathcal{P}(\mathbb{R}^d),\ s \geq 0.$$

Our approach: solve this first!

(nonlinear) superposition principle [Barbu/R.: AOP 2020] ⇓ ⇑ Itô (or Dynkin formula)

McKean-Vlasov SDE (probabilistically weak sense)

$$dX^{s,\zeta}(t) = D(X^{s,\zeta}(t))b(u^{s,\zeta}(t,X^{s,\zeta}(t)))dt + \left(\frac{2\beta(u^{s,\zeta}(t,X^{s,\zeta}(t)))}{u^{s,\zeta}(t,X^{s,\zeta}(t))}\right)^{\frac{1}{2}} dW(t),$$
$$\boldsymbol{\mathcal{L}_{X^{s,\zeta}(t)}(dx) = u^{s,\zeta}(t,x)dx},\ t \geq s \geq 0. \qquad \text{(MVSDE)}$$

Then under suitable conditions on $\beta : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, and $D : \mathbb{R}^d \to \mathbb{R}^d$ (see [Barbu/R.: JFA 2021 and 2023] and [Barbu/R.: Springer LN 2024]) **Corollary I** above applies with

$$\mathcal{P}_0 := \mathcal{P}(\mathbb{R}^d),$$

$$\tilde{\mathcal{P}}_0 = \left\{ u_0(x)dx \,|\, u_0 \geq 0,\, \int_{\mathbb{R}^d} u_0\, dx = 1,\, u_0 \in L^\infty(\mathbb{R}^d; dx) \right\}.$$

Hence the path laws

$$\mathbb{P}_{(s,\zeta)} := \mathcal{L}_{X^{s,\zeta}}, \quad (s,\zeta) \in [0,\infty) \times \mathcal{P}(\mathbb{R}^d),$$

form a nonlinear Markov process.

4.3 FPE = fractional generalized porous media equation

[Barbu/R.: PTRF 2024], [Barbu/da Silva/R.: arXiv: 2308.06388], [Barbu/R.: Springer LN 2024]

Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a **Bernstein function**, e.g. $\Psi(r) = r^s$, $s \in (0,1)$.

Nonlinear fractional Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t} u^{s,\zeta}(t,x) + \boldsymbol{\Psi}(-\boldsymbol{\Delta}_x)(\beta(u^{s,\zeta}(t,x))) + \mathrm{div}_x(D(x)b(u^{s,\zeta}(t,x))u^{s,\zeta}(t,x)) = 0,$$
$$\forall\,(t,x) \in (s,\infty) \times \mathbb{R}^d,\ u^{s,\zeta}(s,x)dx := \zeta \in \mathcal{P}_0,\ s \geq 0. \qquad (\mathrm{FPE}_\psi)$$

Our appro solve first!

(nonlinear) **nonlocal** superposition principle [R./Xie/Zhang: PTRF 2020] $\Downarrow$ $\Uparrow$ Itô (or Dynkin formula)

McKean-Vlasov SDE **with multiplicative Lévy noise** (probabilistically weak sense)

$\exists \mathbb{P}_{(s,\zeta)}$ probability measure on $\mathbb{D}([0,\infty);\mathbb{R}^d)$ solving the martingale problem for $(\mathcal{L}_t, C_c^2(\mathbb{R}^d))$ such that

$$\mathbb{P}_{(s,\zeta)} \circ \boldsymbol{X}(\boldsymbol{t})^{-1}(\boldsymbol{dx}) = \boldsymbol{u}^{s,\zeta}(\boldsymbol{t},\boldsymbol{x})\boldsymbol{dx},\ t \geq s \geq 0. \quad (\mathrm{MVSDE}_\psi)$$

Here

$$\mathcal{L}_t f(x) = b(u^{s,\zeta}(t,x))\, D(x)\, \cdot \nabla f(x) + \frac{\beta(u^{s,\zeta}(t,x))}{u^{s,\zeta}(t,x)}\, p.v. - \int_{\mathbb{R}^d} (f(x+z) - f(x))\nu_{\Psi}(dz)$$

with

$$\nu_{\Psi}(dz) = \left(\int_0^\infty (2t)^{-\frac{d}{2}}\, e^{-\frac{|z|^2}{2t}}\, \mu(dt) \right) dz \text{ and } \Psi(r) = \int_0^\infty (1 - e^{-rt})\mu(dt).$$

Then under suitable conditions on β, b, D and Ψ, **Corollary I** above applies with

$$\tilde{\mathcal{P}}_0 = \mathcal{P}_0 := \left\{ u_0(x)dx \,|\, u_0 \geq 0,\ \int_{\mathbb{R}^d} u_0 dx = 1,\ u_0 \in L^\infty(\mathbb{R}^d; dx) \right\}.$$

Hence

$$\mathbb{P}_{(s,\zeta)}, (s,\zeta) \in [0,\infty) \times \mathcal{P}_0,$$

form a nonlinear Markov process.

4.4 FPE = Burgers equation

[Rehmeier/R.: arXiv: 2212.12424, to appear in JTP]

Nonlinear Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t}u^{s,\zeta}(t,x) - \frac{\partial^2}{\partial x^2}u^{s,\zeta}(t,x) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^{s,\zeta}(t,x)u^{s,\zeta}(t,x)\right) = 0$$

$$\forall\,(t,x) \in (s,\infty)\times\mathbb{R}^1,\ u^{s,\zeta}(s,x)dx := \zeta \in \mathcal{P}_0,\ s \geq 0. \quad \text{(FPE)}$$

Our approach: solve this first!

(nonlinear) superposition principle [Barbu/R.: AOP 2020] $\Downarrow$ $\Uparrow$ Îto (or Dynkin formula)

McKean-Vlasov SDE (probabilistically weak sense)

$$dX^{s,\zeta}(t) = \frac{1}{2}u^{s,\zeta}\left(t, X^{s,\zeta}(t)\right)\mathrm{d}t + \sqrt{2}\,\mathrm{d}W(t)$$

$$\boldsymbol{\mathcal{L}}_{X^{s,\zeta}(t)}(\mathrm{d}\boldsymbol{x}) = \boldsymbol{u}^{s,\zeta}(\boldsymbol{t},\boldsymbol{x})\mathrm{d}\boldsymbol{x},\ t \geq s \geq 0. \qquad \text{(MVSDE)}$$

Then **Corollary I** above applies with $\tilde{\mathcal{P}}_0 = \mathcal{P}_0 =$ as in Example 4.3. Hence the path laws

$$\mathbb{P}_{(s,\zeta)} := \mathcal{L}_{X^{s,\zeta}}\,, \quad (s,\zeta) \in [0,\infty) \times \mathcal{P}_0,$$

form a nonlinear Markov process.

4.5 FPE = 2D vorticity Navier-Stokes equation

[Barbu/R./Zhang: arXiv: 2309.13910, to appear in JEMS]

Nonlinear Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t} u^{s,\zeta}(t,x) + \Delta u^{s,\zeta}(t,x) + \operatorname{div}\Big((k * u^{s,\zeta}(t,\cdot))(x) u^{s,\zeta}(t,x) \Big) = 0$$
$$\forall (t,x) \in (s,\infty) \times \mathbb{R}^2,\ u^{s,\zeta}(s,x)dx := \zeta \in \mathcal{P}(\mathbb{R}^2) \quad \text{(FPE)}$$

Our approach: solve this first!

(nonlinear) superposition principle [Barbu/R.: AOP 2020] ⇓ ⇑ Itô (or Dynkin formula)

McKean-Vlasov SDE (probabilistically weak sense)

$$dX^{s,\zeta}(t) = \Big(k * u^{s,\zeta}(t,\cdot)\Big)\Big(X^{s,\zeta}(t)\Big)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W(t)$$
$$\boldsymbol{\mathcal{L}_{X^{s,\zeta}(t)}(\mathrm{d}x) = u^{s,\zeta}(t,x)\mathrm{d}x},\ t \geq s \geq 0. \quad \text{(MVSDE)}$$

Here

$$k(x) = \frac{(-x_2, x_1)}{2\pi |x|^2_{\mathbb{R}^2}},\ x = (x_1, x_2) \in \mathbb{R}^2. \qquad \text{“Biot-Savart kernel”}$$

Then **Corollary I** applies with

$$\mathcal{P}_0 := \mathcal{P}(\mathbb{R}^2)$$

and

$$\tilde{\mathcal{P}}_0 := \{u_0(x)dx \mid u_0 \geq 0, \int_{\mathbb{R}^d} u_0 dx = 1, \ u_0 \in L^4(\mathbb{R}^d; dx)\}.$$

Hence

$$\mathbb{P}_{s,\zeta} := \mathcal{L}_{X^{s,\zeta}}, \ (s,\zeta) \in [0,\infty) \times \mathcal{P}(\mathbb{R}^2),$$

form a nonlinear Markov process.

Remark

A beautiful result by Sebastian Grube ([PhD-thesis, IRTG 2235, Bielefeld University 2023]) implies that for $(s,\zeta) \in [0,\infty) \times \tilde{\boldsymbol{\mathcal{P}}}_{\mathbf{0}}$ *the weak solution* $X^{s,\zeta}$ *of (MVSDE) above is in fact a strong solution.*