McKean-Vlasov (S)PDEs with additive noise

M. Ottobre
Heriot Watt University and
Maxwell Institute for Mathematical Sciences, Edinburgh

Joint work with
J. Barre', G. Simpson, M. Kolodziejczyk, L. Angeli,
T. Hodgson, D. Crisan, B. Goddard, P. Butta, K. Painter

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KNOWLEDGE MADE USEFUL

Object of interest in this talk

$$\boxed{\partial_t \rho_t(x) = \partial_x \left[V'(x)\rho_t(x) + (F' * \rho_t)(x)\rho_t(x) \right] + \sigma \partial_{xx}\rho_t(x) + Q^{1/2} dW_t}$$

(Stochastic) McKean-Vlasov Equation

Overview

- Motivation

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- Modelling issues
  - Behaviour of the Particle System vs behaviour of the PDE
- Relation to SPDEs - Stochastic McKean-Vlasov equation
  - Can we obtain the SPDE as limit of interacting particles?

McKean-Vlasov equation

- Interacting particle system

$$dX_t^i = -\left[V'(X_t^i) + \frac{1}{N} \sum_{j \neq i} F'(X_t^i - X_t^j) \right] dt + \sqrt{2\sigma} dB_t^i, \quad i = 1, \ldots, N$$

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- Propagation of chaos: each X_t^i converges to X_t solution of a *McKean-Vlasov* SDE

$$dX_t = -V'(X_t) - \left(\int_{\mathbb{R}^d} F'(X_t - y)\rho_t(y)dy\right) dt + \sqrt{2\sigma} dB_t,$$

where $\rho_t = \text{Law}(X_t)$ and it solves (1)

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Particle system and limiting PDE have very different behaviour, in general

- Particle system has a unique invariant measure
- Limiting process undergoes phase transitions, number of invariant measures determined e.g. by noise strength
- In some cases the situation can be even more patological
  - Particle system has periodic behaviour
  - PDE nicely equilibrates to a stationary state

[P. Butta', T. Hodgson, B. Goddard, M.O., K. Painter, Math. Mod. Methods in Appl. Sci, 2022]

Mean-field vs non-mean field

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- Mean field interacting particle system

$$dX_t^i = -\nabla V(X_t^i)dt + \frac{1}{N}\sum_{j\neq i} K(X_t^i - X_t^j)dt + \sqrt{2D}dB_t^i, \quad i = 1, \ldots, N$$

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$$dX_t^i = -\nabla V(X_t^i)dt + \frac{1}{N}\sum_{j\neq i} \mathbf{1}_{(0,R]}(|X_t^{i,N} - X_t^{j,N}|)K(X_t^i - X_t^j)dt + \sqrt{2D}dB_t^i,$$

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- How do we normalise?

$$\frac{1}{N}\sum_{j\neq i} \mathbf{1}_{(0,R]}(|X_t^{i,N} - X_t^{i,N}|)K(X_t^i - X_t^j)$$

or

$$\frac{1}{\#\{j : |X_t^{i,N} - X_t^{i,N}| \leq R\}}\sum_{j=1}^{N} \mathbf{1}_{(0,R]}(|X_t^{i,N} - X_t^{i,N}|)K(|X_t^{i,N} - X_t^{i,N}|)$$

Taking one step back

Finding all the stationary solutions is a difficult problem

$$\partial_x \left[V'(x)\rho(x) + (F' * \rho)(x)\rho(x) \right] + \sigma \partial_{xx}\rho(x) = 0$$

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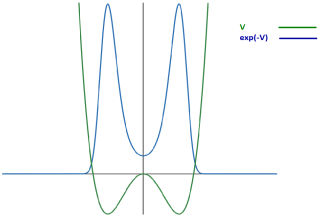
- Number of modes (metastable states) $\sim$ number of (stable) stationary solutions of the ODE



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to count the number of (stable) stationary states of the PDE

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- ....assuming it all goes well

Stochastic McKean-Vlasov

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- Initial Motivation: Tool to find all the stable stationary solutions of the PDE part

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(Stochastic) McKean-Vlasov Equation

- Initial Motivation: Tool to find all the stable stationary solutions of the PDE part
- Can we obtain SPDEs as limits of interacting particles?
  - Series of works of Kurtz et al on derivation of *non-linear* SPDEs with (multiplicative) noise

A couple of preliminary observations

$$\boxed{\rho_t^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i} \stackrel{N\to\infty}{\longrightarrow} \rho_t}$$

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- The evolution (1) preserves mass ((1) is in gradient form)

Common Noise

- Interacting particle system with *common noise*

$$dX_t^i = -\left[V'(X_t^i) + \frac{1}{N}\sum_{j\neq i} F'(X_t^i - X_t^j)\right] dt + \sqrt{2\sigma}\, dB_t^i + \sqrt{2\tilde{\sigma}}\, d\beta_t$$

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- As opposed to (1), equation (2) is *stochastic*

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Some problems...

$$\partial_t \rho_t(x) = \partial_x \left[V'(x)\rho_t(x) + (F' * \rho_t)(x)\rho_t(x) \right] + \sigma \partial_{xx}\rho_t(x) + Q^{1/2} dW_t$$

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- Does not preserve mass (in general)
- Does not have a sign (does not preserve positivity)
- ......cannot expect to obtain it as limit of something of the form $\frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$

The Stochastic McKean-Vlasov equation

Well posedness

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  - The above is good news: setting

$$\mathbb{E}\left[\varphi(\rho_t) | \rho_0 = \rho\right] =: (\mathcal{P}_t\varphi)(\rho) \quad \varphi \in \mathcal{C}_b(L^2; \mathbb{R})$$

does give a semigroup

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- Existence and uniqueness in L^2 for trace class noise: a combination of Burgers' equation + McKean Vlasov PDE arguments

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    does give a semigroup
- Existence and uniqueness in L^2 for trace class noise: a combination of Burgers' equation + McKean Vlasov PDE arguments
  - Existence and uniqueness for cylindrical noise not known yet

The Stochastic McKean-Vlasov equation

Ergodicity

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- Depending on the choice of V, F, σ the McKean-Vlasov PDE, i.e. the evolution

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  has *multiple* stationary solutions (equilibria)

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  has *multiple* stationary solutions (equilibria)
- Adding noise restores uniqueness of the invariant measure (equilibrium)
  - with a caveat .... there exists *at most one* invariant measure of the SMKV equation. *Existence* still to be done.

Weighted Particle system converging to SMKV

- Start with simpler noise

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$Q : \mathbb{T} \to \mathbb{R}$ and W_t one-dimensional Brownian Motion

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- Ansatz:

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$$dA_t^i = \frac{Q(X_t^i)}{\frac{1}{N} \sum_j \phi_\epsilon (X^i - X^j)} dW_t$$

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- Claim :

$$\rho_t^N : \frac{1}{N} \sum_{i=1}^{N} A_t^i \delta_{X_t^i} \overset{N \to \infty}{\longrightarrow} \rho_t$$

i.e. $\lim_{N \to \infty} \langle \rho_t^N - \rho_t, f \rangle = 0$, for every test function f.

Explaining the Ansatz in simplified setting: SHE

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Explaining the Ansatz in simplified setting: SHE

- Start with simpler noise

$$\partial_t \rho_t(x) = \sigma \partial_{xx} \rho_t(x) + Q(x) dW_t$$

  $Q : \mathbb{T} \to \mathbb{R}$ and W_t one-dimensional Brownian Motion

- Ansatz:

$$dX_t^i = \sqrt{2\sigma} dB_t^i$$

$$dA_t^i = \frac{Q(X_t^i)}{\frac{1}{N} \sum_j \phi_\epsilon(X^i - X^j)} dW_t$$

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- Claim :

$$\rho_t^N : \frac{1}{N}\sum_{i=1}^N A_t^i \delta_{X_t^i} \overset{N\to\infty}{\longrightarrow} \rho_t$$

i.e. $\lim_{N\to\infty} \langle \rho_t^N - \rho_t, f\rangle = 0,$ for every test function f.

Explaining the ansatz continued

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- Ito formula

$$d(A_t^i f(X_t^i)) = A_t^i f'(X_t^i)\sqrt{2\sigma}\, dB_t^i + \sigma A_t^i f''(X_t^i) dt + f(X_t^i)\frac{Q(X_t^i)}{\frac{1}{N}\sum_j \phi_\epsilon(X^i - X^j)} dW_t$$

Explaining the ansatz continued

- Take sums

$$d\left[\frac{1}{N}\sum_{i=1}^{N}(A_t^i f(X_t^i))\right] = \frac{1}{N}\sum_{i=1}^{N} A_t^i f'(X_t^i)\sqrt{2\sigma}dB_t^i + \sigma\frac{1}{N}\sum_{i=1}^{N} A_t^i f''(X_t^i)dt$$

$$+ \frac{1}{N}\sum_{i=1}^{N} f(X_t^i)\frac{Q(X_t^i)}{\frac{1}{N}\sum_j \phi_\epsilon(X^i - X^j)}dW_t$$

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$$+\frac{1}{N}\sum_{i=1}^{N} f(X_t^i)\frac{Q(X_t^i)}{\frac{1}{N}\sum_j \phi_\epsilon(X^i - X^j)}dW_t$$

- For less than the martingale term,

$$d(\rho_t^{N,\epsilon}, f) \simeq \sigma(\rho_t^{N,\epsilon}, f'') + \left(\frac{fQ}{\eta_t^{N,\epsilon} * \phi_\epsilon}, \eta_t^{N,\epsilon}\right)dW_t \quad \eta_t^{N,\epsilon} = \frac{1}{N}\sum \delta_{X_t^i}$$
$$\xrightarrow{N\to\infty} \sigma(\rho_t^{\epsilon}, f'') + \left(\frac{fQ}{\eta_t^{\epsilon} * \phi_\epsilon}, \eta_t^{\epsilon}\right)dW_t$$
$$\xrightarrow{\epsilon\to 0} \sigma(\rho_t, f'') + \left(\frac{fQ}{\eta_t}, \eta_t\right)dW_t$$
$$= \sigma(\rho_t, f'') + (f, Q)dW_t$$

Something on the proof - fixing the noise

- Initial interest on

$$\partial_t \rho_t(x) = \partial_x \left[V'(x)\rho_t(x) + (F' * \rho_t)(x)\rho_t(x) \right] + \sigma \partial_{xx}\rho_t(x) + Q(x) dW_t$$

Something on the proof - fixing the noise

- Initial interest on

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- Results for perturbations of type

$$\partial_t \rho_t(x) = \partial_x \left[V'(x)\rho_t(x) + (F' * \rho_t)(x)\rho_t(x)\right] + \sigma \partial_{xx}\rho_t(x) + \mathcal{U}_t$$

  where $\mathcal{U}_t$ is

  - a piecewise smooth deterministic path
  - a $\alpha-$ Hölder continuous deterministic path (fix the noise)
  - a finite sum of Brownian Motions
  - a trace class Wiener process

Why is it convenient to fix the noise?

- The empirical measure $\mu_t^N := \frac{1}{N}\sum_j \delta_{(X_t^j, A_t^j)}$ converges to $\mu_t = \mu_t(x,a)$ which solves

$$\partial_t \mu_t = \partial_{xx}\mu_t + \partial_x \left[(V' + \int aF'(x-y)\mu_t(dy,da))\mu_t \right] - \frac{Q}{\Phi_\epsilon * \zeta_t}\partial_a \mu_t dW_t + \frac{1}{2}\frac{Q^2}{(\Phi_\epsilon * \zeta_t)^2}\partial_{aa}\mu_t$$

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- Fixing the noise is ‘natural’ as the noise we fix is the ‘common noise’ on the weights

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