Sharp upper bounds on hitting probabilities for the solution to the stochastic heat equation on the line

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Based on joint work with:

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Overview

- Introduction to the problems of hitting probabilities
- State of the art for Gaussian random fields
- Typical results and methods for non-linear systems of s.p.d.e.'s
- Sharp upper (and lower) bounds for the non-linear random string

Hitting probabilities and polarity of points for random fields

Let $U = (U(x),\ x \in \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous stochastic process.

Fix $I \subset \mathbb{R}^k$, compact with positive Lebesgue measure.

The range of U over I is the random compact set

$$U(I) = \{U(x),\ x \in I\}.$$

Question. (Hitting probabilities) For $A \subset \mathbb{R}^d$, what are bounds on the probability that U hits A, that is,

$$\mathbb{P}\{U(I) \cap A \neq \emptyset\}\,?$$

Related question. (Polarity of points) Fix $z \in \mathbb{R}^d$. Does U fail to hit z, that is, do we have

$$\mathbb{P}\{\exists x \in I : U(x) = z\} = 0\,?$$

Polarity. If $\mathbb{P}\{\exists x \in I : U(x) = z\} = 0$, then z is polar for U.

Typically, there is a critical value $Q(k)$ such that:

- if $d < Q(k)$, then points are not polar
- if $d > Q(k)$, then points are polar
- at the critical valued $d = Q(k)$: ???

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First example: the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}_+^k)$ denote an k-parameter $\mathbb{R}^d$-valued Brownian sheet, that is, a centered continuous Gaussian random field

$$W(x) = (W_1(x), \ldots, W_d(x))$$

with covariance

$$E[W_i(x)W_j(y)] = \delta_{i,j} \prod_{\ell=1}^{k} \min(x_\ell, y_\ell), \qquad i,j \in \{1, \ldots, d\},$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$.

Theorem 1 (Khoshnevisan and Shi, 1999)

Fix $M > 0$. Let I be a box. There exists $0 < C < \infty$ such that for all compact sets $A \subset B(0, M)$ $(\subset \mathbb{R}^d)$,

$$\frac{1}{C}\ Cap_{d-2k}(A) \leqslant \mathbb{P}\{W(I) \cap A \neq \emptyset\} \leqslant C\ Cap_{d-2k}(A).$$

Example. $A = \{z\}$.

$$\text{Cap}_{d-2k}(\{z\}) = \begin{cases} 1 & \text{if } d < 2k, \\ 0 & \text{if } d \geqslant 2k, \end{cases}$$

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Let $(V(x),\ x \in \mathbb{R}^k)$ be centered continuous Gaussian, values in $\mathbb{R}^d$, i.i.d. components: $V(x) = (V_1(x), \dots, V_d(x))$. Let I be a box. Assume:

(C) There exists $0 < c < \infty$ and $H_1, \dots, H_k \in]0,1[$ such that for all $x, y \in I$,

$$c^{-1} \sum_{\ell=1}^{k} |x_\ell - y_\ell|^{H_\ell} \leqslant \Delta(x,y) := \|V_1(x) - V_1(y)\|_{L^2(\Omega)} \leqslant c \sum_{\ell=1}^{k} |x_\ell - y_\ell|^{H_\ell}$$

and some non-degeneracy assumptions.

Theorem 2 (Biermé, Lacaux & Xiao, 2007)

Fix $M > 0$. Set $Q = \sum_{\ell=1}^{k} \frac{1}{H_\ell}$. Then there is $0 < C < \infty$ such that for every compact set $A \subset B(0,M)$,

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Example. The Brownian sheet. Theorem 2 is close to Theorem 1: for $\ell = 1, \dots, k$, $x_\ell \mapsto W(x_1, \dots, x_\ell, \dots, x_k)$ is a Brownian motion, so $H_\ell = \frac{1}{2}$ and

$$Q = \sum_{\ell=1}^{k} \frac{1}{H_\ell} = 2k$$

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Main difference between Theorems 2 and 1

In both theorems, the "dimension" that appears is $d - Q = d - 2k$;

For the Brownian sheet, both theorems identify the critical dimension $d = 2k$.

But compare the right-hand sides:

in Theorem 2, Hausdorff measure.
in Theorem 1, capacity.

Case where $A = \{z\}$:

$$\mathrm{Cap}_{d-Q}(\{z\}) = \begin{cases} 1 & \text{if } d < Q, \\ 0 & \text{if } d = Q, \\ 0 & \text{if } d > Q, \end{cases} \qquad \mathcal{H}_{d-Q}(\{z\}) = \begin{cases} \infty & \text{if } d < Q, \\ 1 & \text{if } d = Q, \\ 0 & \text{if } d > Q. \end{cases}$$

If $d = Q$, Theorem 2 says: $0 \leqslant \mathbb{P}\{\exists x \in I : V(x) = z\} \leqslant 1$ (not informative)!

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Funaki's random string

Let $(V(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ be an $\mathbb{R}^d$-valued random field such that

$$\frac{\partial}{\partial t} V(t,x) = \frac{\partial^2}{\partial x^2} V(t,x) + \dot{W}(t,x), \qquad x \in \mathbb{R},\ t > 0,$$

$V(0,\cdot) : \mathbb{R} \to \mathbb{R}^d$ given, $\dot{W}(t,x)$ is $\mathbb{R}^d$-valued space-time white noise (Gaussian).

Theorem 3 (Mueller & Tribe, 2002)

The critical dimension for hitting points is $d = 6$ and points are polar in this dimension.

(Also treat the issue of double points for this random field)

Theorem 4 (D., Khoshnevisan & E. Nualart, 2007)

Fix $M > 0$. There is $0 < C < \infty$ such that for every compact set $A \subset B(0,M)$,

$$C^{-1}\ \mathrm{Cap}_{d-6}(A) \leqslant \mathbb{P}\{V(I) \cap A \neq \emptyset\} \leqslant C \mathcal{H}_{d-6}(A).$$

$$\left(Q = \frac{1}{\frac{1}{4}} + \frac{1}{\frac{1}{2}} = 6\right)$$

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Systems of nonlinear wave equations in spatial dimension 1

Let $(U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ be an $\mathbb{R}^d$-valued random field such that

$$\frac{\partial^2}{\partial t^2} U(t,x) = \frac{\partial^2}{\partial x^2} U(t,x) + \sigma(U(t,x))\dot{W}(t,x), \qquad x \in \mathbb{R},\ t > 0,$$

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$$\frac{1}{C}\ Cap_{d-4}(A) \leqslant \mathbb{P}\{U(I) \cap A \neq \emptyset\} \leqslant C\ Cap_{d-4}(A).$$

$$\left(Q = \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{2}} = 4\right)$$

The critical dimension for hitting points is $d = 4$ and points are polar in this dimension.

The proof uses Malliavin calculus (lower bound) and Cairoli's maximal inequality for multi-parameter martingales (upper bound).

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Other non-linear systems of stochastic p.d.e.'s

Let L be a partial differential operator (e.g. $L = \frac{\partial}{\partial t} - \Delta$ or $L = \frac{\partial^2}{\partial t^2} - \Delta$).

Let $U(t,x) = (U^1(t,x), \ldots, U^d(t,x)) \in \mathbb{R}^d$ be the solution of

$$\begin{cases} LU^1(t,x) &= b^1(U(t,x)) + \sum_{j=1}^d \sigma_{1,j}(U(t,x))\dot{W}_j(t,x), \\ & \vdots \\ LU^d(t,x) &= b^d(U(t,x)) + \sum_{j=1}^d \sigma_{d,j}(U(t,x))\dot{W}_j(t,x), \end{cases}$$

$$t \in]0,T], \quad x \in \mathbb{R}^k.$$

smooth (Lipschitz) non-linearities: $b^i,\ \sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}, \qquad i = 1, \ldots, d$

Initial conditions: e.g. $U(0,x) = U_0(x)$ given.

$\dot{W}_j(t,x)$: independent space-time white noises, real-valued.

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Suppose that we have optimal Hölder exponents for the solution:

$$c(p)\,\Delta(t,x;s,y) \leqslant \|U(t,x)-U(s,y)\|_{L^p(\Omega)} \leqslant C(p)\,\Delta(t,x;s,y),$$

where

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Define

$$Q = \sum_{\ell=0}^{k} \frac{1}{H_\ell}.$$

Typical result 1

Fix $\eta > 0$. Then

$$c_\eta\, Cap_{d-Q+\eta}(A) \leqslant \mathbb{P}\{U(I\times J)\cap A \neq \emptyset\} \leqslant C_\eta \mathcal{H}_{d-Q-\eta}(A)$$

Remarks. (a) This is similar to the result of Biermé, Lacaux and Xiao (2007).
(b) In the critical dimension $d = Q$, this is not informative!
(c) There is an additional parameter η on the left- and right-hand sides: this is less sharp than in the Gaussian case.

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The nonlinear system of stochastic heat equations

$(U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$: $\mathbb{R}^d$-valued random field such that for $t > 0$, $x \in \mathbb{R}$,

$$\frac{\partial}{\partial t} U(t,x) = \frac{\partial^2}{\partial x^2} U(t,x) + b(U(t,x)) + \sigma(U(t,x)) \cdot \dot{W}(t,x),$$

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$\sigma = (\sigma_{i,j},\ i,j = 1,\dots,d) : \mathbb{R}^d \to \mathbb{M}_{d\times d}$, $b = (b_i,\ i = 1,\dots,d) : \mathbb{R}^d \to \mathbb{R}^d$

Assumption. The $\sigma_{i,j}$ and b_i are C^∞, bounded, with bounded derivatives of all orders, and σ is uniformly elliptic.

Theorem 6 (D., Khoshnevisan & E. Nualart, 2009)

Fix $\eta > 0$, $M > 0$ and two non-trivial compact intervals I and J. There exists $c > 0$ such that for all compact sets $A \subseteq [-M, M]^d$,

$$c^{-1}\,\mathrm{Cap}_{d-6+\eta}(A) \leqslant \mathbb{P}\{U(I\times J) \cap A \neq \emptyset\} \leqslant c\,\mathcal{H}_{d-6-\eta}(A).$$

The nonlinear system of stochastic heat equations

$(U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$: $\mathbb{R}^d$-valued random field such that for $t > 0$, $x \in \mathbb{R}$,

$$\frac{\partial}{\partial t} U(t,x) = \frac{\partial^2}{\partial x^2} U(t,x) + b(U(t,x)) + \sigma(U(t,x)) \cdot \dot{W}(t,x),$$

$U(0,\cdot) : \mathbb{R} \to \mathbb{R}^d$ given, $\dot{W}(t,x)$ is $\mathbb{R}^d$-valued space-time white noise

$\sigma = (\sigma_{i,j},\ i,j = 1,\dots,d) : \mathbb{R}^d \to \mathbb{M}_{d\times d}$, $b = (b_i,\ i = 1,\dots,d) : \mathbb{R}^d \to \mathbb{R}^d$

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Proving the upper bound

Let $U = (U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous random field.

Typical result 2

Let $D \subset \mathbb{R}^d$. In addition to knowing the Hölder exponents H_ℓ, assume that for any $t > 0$ and $x \in \mathbb{R}^k$, $U(t,x)$ has a density $p_{(t,x)}$, and

$$\sup_{z \in D^{(2)}} \sup_{(t,x) \in (I \times J)^{(1)}} p_{(t,x)}(z) \leq C \tag{1}$$

($D^{(2)}$ is the 2-enlargement of D.)

Then for any $\eta > 0$, for every Borel set $A \subset D$,

$$P\{U(I) \cap A \neq \emptyset\} \leq C \mathcal{H}_{d-\eta-Q}(A).$$

Remarks. (a) Gaussian case: (1) becomes $\det \operatorname{Cov}(U(t,x), U(t,x)) > 0$.

(b) Non-Gaussian case: Condition (1) can often be obtained by using Malliavin calculus.

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Lower bounds

Let $U = (U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous random field.

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In addition to knowing the Hölder exponents H_ℓ, assume that:

- *the density of $U(t,x)$ is strictly positive*
- *the two-point density of $(U(s,y), U(t,x))$ satisfies the upper bound*

$$p_{s,y;t,x}(z_1,z_2) \leqslant [\Delta(s,y;t,x)]^{-(d+\eta)} \exp\left[-\frac{\|z_1 - z_2\|^2}{c\,\Delta^2(s,y;t,x)}\right].$$

These two properties (obtained via Malliavin calculus) imply the lower bound

$$P\{U(I \times J) \cap A \neq \emptyset\} \geq c\ \mathrm{Cap}_{d+\eta-Q}(A)$$

where $Q = \sum_{\ell=0}^{k} \frac{1}{H_\ell}$. (optimal if $\eta = 0$)

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Improving the lower bound

Question. Is the extra $\eta > 0$ needed in the nonlinear case?

Theorem 7 (Fei Pu, Thesis, 2018; D. & Pu, 2021)

In Theorem 6 (nonlinear system of stochastic heat equations), it is possible to remove the η in the lower bound:

$$c^{-1}\,\mathrm{Cap}_{d-6}(A) \leqslant \mathbb{P}\{U(I \times J) \cap A \neq \emptyset\}.$$

The proof consists in refining the Malliavin calculus argument used for Theorem 6.

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Improving the upper bound

Proofs of upper bound use a covering argument:

$I \times J \subset (0, T] \times [-M, M]$. For $n, m, \ell \in \mathbb{N}$, set

$$t_m^n := m\, 2^{-nH_1^{-1}}, \qquad x_\ell^n := \ell\, 2^{-nH_\ell^{-1}},$$

and

$$I_m^n = [t_m^n, t_{m+1}^n], \qquad J_\ell^n = [x_\ell^n, x_{\ell+1}^n], \qquad R_{m,\ell}^n = I_m^n \times J_\ell^n.$$

For $z \in \mathbb{R}^d$, need a good estimate of

$$\mathbb{P}\left\{ \inf_{(t,x)\in R_{m,\ell}^n} |U(t,x) - z| \leqslant 2^{-n} \right\}.$$

Reverse triangle inequality: this is bounded above by

$$\mathbb{P}\left\{ |U(t_m^n, x_\ell^n) - z| \leqslant 2^{-n} + \sup_{(t,x)\in R_{m,\ell}^n} |U(t,x) - U(t_m^n, x_\ell^n)| \right\}.$$

This can come from bounds on the joint probability density function of

$$(F_1, F_2) := \left(U(t_m^n, x_\ell^n), \sup_{(t,x)\in R_{m,\ell}^n} (U(t,x) - U(t_m^n, x_\ell^n)) \right)$$

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Improving the upper bound: the Gaussian case

$$(F_1, F_2) = \left(U(t_m^n, x_\ell^n), \sup_{(t,x) \in R_{m,\ell}^n} (U(t,x) - U(t_m^n, x_\ell^n)) \right)$$

Notice that

$$F_2 = Z_{1,1} + Z_{1,2}$$

where

$$Z_{1,1} := \sup_{(t,x) \in R_{m,\ell}^n} (U(t,x) - \mathbb{E}[U(t,x) \mid U(t_m^n, x_\ell^n)]),$$

$$Z_{1,2} := \sup_{(t,x) \in R_{m,\ell}^n} (\mathbb{E}[U(t,x) \mid U(t_m^n, x_\ell^n)] - U(t_m^n, x_\ell^n)).$$

Then

$$(F_1, F_2) \sim (F_1, Z_{1,1}, Z_{1,2}) \sim (F_1, Z_{1,1}, 2^{-n} F_1)$$

and F_1 and $Z_{1,1}$ are independent because U is Gaussian.

This argument does not carry over to the non-Gaussian case.

Improving the upper bound: the non-Gaussian case

Decoupled system:

$$U(t,x) = (U^1(t,x), \dots, U^d(t,x)),$$

and the U^i are i.i.d. copies of $u(t,x)$, where

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + b(u(t,x)) + \sigma(u(t,x))\, \dot{W}(t,x), \qquad (2)$$

$u(0,\cdot) : \mathbb{R} \to \mathbb{R}$ given, $\dot{W}(t,x)$ is real-valued space-time white noise,

$\sigma, b : \mathbb{R} \to \mathbb{R}$.

Problem. For (t_0, x_0) fixed, give bounds on the joint probability density function of

$$F_1^u = u(t_0, x_0) \qquad \text{and} \qquad F_2^u(\zeta_1, \zeta_2) = \sup_{\substack{t_0 \le t \le t_0+\zeta_1 \\ x_0 \le x \le x_0+\zeta_2}} (u(t,x) - u(t_0,x_0)).$$

Have not been able to do this.

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Have not been able to do this.

Solving the problem

Recall that

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(u(t,x))\, \dot{W}(t,x) + b(u(t,x)),$$

Consider the Gaussian process $(v(t,x))$ such that

$$\begin{cases} \partial_t v(t\,,x) = \frac{1}{2}\partial_x^2 v(t\,,x) + \dot{W}(t\,,x), & t > 0,\ x \in \mathbb{R}, \\ v(0) \equiv 0. \end{cases} \tag{3}$$

(same $\dot{W}$ as in (2)). Define

$$F_2^v(\zeta_1, \zeta_2) = \sup_{\substack{t_0 \le t \le t_0 + \zeta_1 \\ x_0 \le x \le x_0 + \zeta_2}} (v(t,x) - v(t_0, x_0)).$$

Then:

(a) Have obtained good bounds on the density of $(F_1^u, F_2^v(\zeta_1, \zeta_2))$;

(b) Can show that $L_{t_0,x_0}(\zeta_1, \zeta_2)$ is small, where

$$\sup_{\substack{t_0 \le t \le t_0 + \zeta_1 \\ x_0 \le x \le x_0 + \zeta_2}} |u(t,x) - u(t_0,x_0) - \sigma(u(t_0,x_0))\,(v(t,x) - v(t_0,x_0)|; \tag{4}$$

(c) these properties are sufficient to establish the sharp upper bound

$$\mathbb{P}\{U(I \times J) \cap A \neq \emptyset\} \leqslant C\, \mathcal{H}_{d-6}(A).$$

Solving the problem

Recall that

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(u(t,x))\, \dot{W}(t,x) + b(u(t,x)),$$

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Precise statements

Theorem 8 (D., Fei Pu & David Nualart, 2025)

(a) There is a constant $c = c(I, J)$ *such that: for* $(t_0, x_0) \in I \times J$ *and* $\zeta_1, \zeta_2 \in]0, 1[$, *the density* $p_{\zeta_1,\zeta_2}(z_1, z_2)$ *of* $F = (F_1^u, F_2^v(\zeta_1, \zeta_2))$, $z_1 \in \mathbb{R}$, $z_2 > 0$, *is such that, for small* $\zeta_1 > 0$ *and* $\zeta_2 > 0$, $z_1 \in \mathbb{R}$ *and* $z_2 \geq \zeta := \max(\zeta_1^{1/4}, \zeta_2^{1/2})$,

$$p_{\zeta_1,\zeta_2}(z_1, z_2) \leq \frac{c}{\zeta} \exp\left(-\frac{z_2^2}{c\,\zeta^2}\right).$$

(b) Let $L_{t_0,x_0}(\zeta_1, \zeta_2)$ *be as in* (4)*. Then for all large* $k \in \mathbb{N}$,

$$\|L_{t_0,x_0}(\zeta_1, \zeta_2)\|_{L^k(\Omega)} \leqslant \zeta^{\frac{3}{2}}$$

(c) Suppose that $\sigma \in C^3(\mathbb{R})$*, Lipschitz, bounded and* $\inf_{z\in\mathbb{R}} \sigma(z) > 0$*. Then there exists* $C = C(I, J) < \infty$ *such that for all compact sets* $A \subset \mathbb{R}^d$,

$$\mathbb{P}\{U(I \times J) \cap A \neq \emptyset\} \leqslant C\,\mathcal{H}_{d-6}(A).$$

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