Parabolic rescaling of a stochastic wave map: limit and fluctuations

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joint work with M. Xie

Irregular Stochastic Analysis

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The equation

We start from the following stochastic damped wave equation

$$\begin{cases} \partial_t^2 u(t,x) + |\partial_t u(t,x)|^2 u(t,x) = \partial_x^2 u(t,x) + |\partial_x u(t,x)|^2 u(t,x) \\ \qquad\qquad -\gamma\, \partial_t u(t,x) + \big(u(t,x) \times \partial_t u(t,x)\big) \circ \partial_t w(t,x), \\ u(0,x) = u_0(x), \qquad \partial_t u(0,x) = v_0(x), \end{cases} \tag{1}$$

in dimension $1+1$, whose solution takes value in $\mathbb{S}^2$, the unitary sphere of $\mathbb{R}^3$.

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in dimension $1+1$, whose solution takes value in $\mathbb{S}^2$, the unitary sphere of $\mathbb{R}^3$.

Here γ is a positive constant friction coefficient and the initial condition (u_0, v_0) is taken in $\mathcal{M}$, where

$$\mathcal{M} := \big\{(u,v) : \mathbb{R} \mapsto T\mathbb{S}^2\big\},$$

and

$$T\mathbb{S}^2 := \big\{(h,k) \in \mathbb{S}^2 \times \mathbb{R}^3 \,:\, h \cdot k = 0\big\}.$$

The noise

Here the stochastic differential is in Stratonovich form and

$w(t),\ t \geq 0$, is a spatially homogeneous Wiener process on a complete filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$,

We assume that the spectral measure μ of the noise has a density m such that

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$$\int_{\mathbb{R}} (1 + x^2)\, m(x)\, dx < \infty.$$

The well-posedness of equations of this type have been already studied in the existing literature.

See the works by Brzezniak- Ondrejat and others.

Our problem

We have studied equation (1) under a parabolic rescaling,

which transforms the system into a family of equations parametrized by a small parameter $\epsilon > 0$, in which time is dilated and space is rescaled.

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which transforms the system into a family of equations parametrized by a small parameter $\epsilon > 0$, in which time is dilated and space is rescaled.

Namely, for every $\epsilon > 0$ we define

$$u_\epsilon(t, x) := u(t/\epsilon, x/\sqrt{\epsilon}), \qquad (t, x) \in [0, +\infty) \times \mathbb{R},$$

and

investigate the asymptotic behavior of u_ϵ, as $\epsilon \downarrow 0$.

In particular, we study the transition from the stochastic hyperbolic regime to a deterministic parabolic limit.

The deterministic case

This problem has been already addressed by Jiang, Luo, Tang, and Zarnescu (2019) when there is no noise, in the more delicate situation of space dimension $d > 2$.

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In that case, in order to get the same convergence, which is clearly only local in time, the authors expanded $u_\epsilon(t,x)$ as

$$u_\epsilon(t,x) = u(t,x) + u_0^I(t/\epsilon, x) + \sqrt{\epsilon}\, u_R^\epsilon(t,x),$$

where u_0^I is a suitable boundary layer which is given explicitly and u_R^ϵ is the solution of the damped wave equation

$$\epsilon\, \partial_t^2 u_R^\epsilon(t,x) = \partial_x^2 u_R^\epsilon(t,x) - \partial_t u_R^\epsilon(t,x) + S(u_R^\epsilon)(t,x) + R(u_R^\epsilon)(t,x),$$

for some singular term $S(u_R^\epsilon)$ and regular term $R(u_R^\epsilon)$.

The rescaled equation

After converting the Stratonovich's differential into the Itô's one, u_ϵ satisfies the equation

$$\left\{\begin{array}{l} \epsilon\partial_t^2 u(t,x) + \epsilon|\partial_t u(t,x)|^2 u(t,x) = \partial_x^2 u(t,x) + |\partial_x u(t,x)|^2 u(t,x) \\ \qquad\qquad -\gamma_0\, \partial_t u(t,x) + \sqrt{\epsilon}\big(u(t) \times \partial_t u(t)\big)\partial_t w^\epsilon(t,x), \\ u(0,x) = u_0^\epsilon(x), \qquad \partial_t u(0,x) = v_0^\epsilon(x), \end{array}\right. \tag{2}$$

for some initial conditions u_0^ϵ and v_0^ϵ depending on ϵ. Notice that here the friction γ is enhanced by the new one

$$\gamma_0 := \gamma + \frac{1}{2}\,\mu(\mathbb{R}).$$

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The rescaled noise $w^\epsilon(t,x)$ is white in time and inherits a spatial covariance structure adapted to the parabolic scaling.

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it would be possible to introduce an Itô noise instead, and still preserve the geometric constraint of remaining on $\mathbb{S}^2$.

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- However, this alternative would require imposing a sufficiently large friction coefficient γ, in order to control the dynamics and study the limiting behavior of u_ϵ, instead of any arbitrary $\gamma > 0$, as here.

- Moreover, the limiting equation would forget about the noise used in the hyperbolic system.

Some related previous work

We have already studied similar asymptotic problems for stochastic damped wave equations with constraints.

However, in those works we considered equations on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, where the solutions were constrained to lie on the Hilbert manifold of functions in $H := L^2(\mathcal{O})$ with norm equal 1.

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The analysis of both papers

- was motivated by the study of the small-mass limit, also known as Smoluchowski-Kramers approximation;
- was only related to the study of the limit of u_ϵ.

The first model

Together with Z. Brzeźniak we have studied

$$
\begin{cases}
\epsilon\,\partial_t^2 u_\epsilon(t,x) + \epsilon|\partial_t u_\epsilon(t)|_H^2 u_\epsilon(t,x) = \Delta u_\epsilon(t,x) \\
\qquad\qquad + |\nabla u_\epsilon(t)|_H^2 u_\epsilon(t,x) - \gamma\,\partial_t u_\epsilon(t,x) + \sigma(u_\epsilon(t))\,\partial_t w(t,x), \\
u_\epsilon(0,x) = u_{\text{in}}(x), \quad \partial_t u_\epsilon(0,x) = v_{\text{in}}(x), \\
u_\epsilon(t,x) = 0, \quad x \in \partial\mathcal{O},
\end{cases}
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when the parameter $\epsilon \downarrow 0$. Here $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded and regular domain and σ has a suitable form.

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when the parameter $\epsilon \downarrow 0$. Here $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded and regular domain and σ has a suitable form.

We have shown that

the solution u_ϵ converges to the solution of a stochastic parabolic equation subject to the same constraint, where an extra noise-induced drift emerges.

The small-mass limit

We have shown that under suitable regularity conditions for the initial conditions, x for every $T > 0$ and $\eta > 0$ we have

$$\lim_{\epsilon \to 0} \mathbb{P}\left(|u_\epsilon - u|_{L^4(0,T;H^1)} > \eta\right) = 0,$$

where $u \in L^2(\Omega; L^4(0,T;H^1 \cap M) \cap L^2(0,T;H^2))$ is the unique solution of the equation

$$\left\{\begin{array}{l} \gamma\, \partial_t u(t,x) = \Delta u(t,x) + |\nabla u(t)|_H^2 u(t,x) - \dfrac{1}{2}\|\sigma(u(t))\|^2_{\mathcal{L}_2(K,H)} u(t) \\[10pt] \qquad\qquad\qquad\qquad +\sigma(u(t))\partial_t w(t,x), \\[10pt] u(0,x) = u_{\text{in}}(x), \qquad u(t,x) = 0, \quad x \in \partial\mathcal{O}. \end{array}\right.$$

A second model

Together with M. Xie, we have considered the constrained system

$$\left\{\begin{aligned} &\epsilon\,\partial_t^2 u_\epsilon(t,x) + \epsilon\,|\partial_t u_\epsilon(t)|_H^2\, u_\epsilon(t,x),\\ &\qquad = \Delta u_\epsilon(t,x) + |\nabla u_\epsilon(t)|_H^2\, u_\epsilon(t,x) - \gamma\,\partial_t u_\epsilon(t,x)\\ &\qquad\qquad + \sqrt{\epsilon}\,\big(u_\epsilon(t) \times \partial_t u_\epsilon(t)\big) \circ \partial_t w(t,x), \end{aligned}\right. \tag{3}$$

with Dirichlet boundary conditions and initial conditions

$$u_\epsilon(0,x) = u_{\text{in}}(x), \quad \partial_t u_\epsilon(0,x) = v_{\text{in}}(x),$$

in a bounded interval $[0,L]$. Here

$$u : [0,+\infty) \times [0,L] \to \mathbb{R}^3.$$

The noise

The noise $w(t,x)$ is a scalar cylindrical Wiener process given by

$$w(t,x) = \sum_{i=1}^{\infty} \xi_i(x)\beta_i(t),$$

for a sequence of independent Brownian motions defined on some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$.

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We assume that $\xi_i \in C^2([0,L])$, for all $i \in \mathbb{N}$. Moreover, the functions

$$\varphi(x) := \sum_{i=1}^{\infty} |\xi_i(x)|^2, \quad \varphi_1(x) := \sum_{i=1}^{\infty} |\xi_i'(x)|^2, \quad \varphi_2(x) := \sum_{i=1}^{\infty} |\xi_i''(x)|^2,$$

belong to $C([0,L])$. In particular, $\varphi \in C^2([0,L])$.

The limiting result

We have proved that if the initial condition is sufficiently regular, then for every $T > 0$ and $\delta < 2$

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |u_\epsilon(t) - u(t)|_{H^\delta} > \eta\right) = 0,$$

for every $\eta > 0$, where

$$u \in C([0,+\infty); H^2), \qquad \partial_t u \in L^2(0,+\infty; H)$$

is the unique solution of the equation

$$\begin{aligned} \partial_t\left[\left(\gamma + \frac{1}{2}\varphi\,|u|^2_{\mathbb{R}^3}\right)u\right] &= \Delta u + \frac{3\varphi}{2\gamma}(\Delta u \cdot u)u \\ &\quad + \left[1 + \frac{3\varphi}{2\gamma}|u|^2_{\mathbb{R}^3}\right]|\nabla u|^2_H u, \end{aligned} \tag{4}$$

with Dirichlet b.c.

We emphasize that the methods we developed for those two models

- cannot be reduced to a parabolic rescaling,
- relied on substantially different techniques,

due to the fundamentally distinct nature of the imposed functional constraints.

Back to our problem

With M. Xie we have considered the equation

$$\left\{\begin{array}{l} \epsilon\partial_t^2 u(t,x) + \epsilon|\partial_t u(t,x)|^2 u(t,x) = \partial_x^2 u(t,x) + |\partial_x u(t,x)|^2 u(t,x) \\[10pt] \qquad\qquad -\gamma_0\, \partial_t u(t,x) + \sqrt{\epsilon}\big(u(t)\times\partial_t u(t)\big)\partial_t w^\epsilon(t,x), \\[10pt] u(0,x) = u_0^\epsilon(x), \qquad \partial_t u(0,x) = v_0^\epsilon(x), \end{array}\right. \tag{5}$$

where, we recall,

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where, we recall,

$$\gamma_0 := \gamma + \frac{1}{2}\,\mu(\mathbb{R}).$$

In 2007, Brzeźniak and Ondreijat showed that for every $\epsilon > 0$ such equation is well-posed in

$$\big(H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^3) \times H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^3)\big) \cap \mathcal{M}.$$

Well-posedness

In fact, we could improve such result and show that

for every $T > 0$ and $(u_0^\epsilon, v_0^\epsilon) \in \left(\dot{H}^2(\mathbb{R};\mathbb{R}^3) \times H^1(\mathbb{R};\mathbb{R}^3)\right) \cap \mathcal{M}$, there exists a unique global strong adapted solution u_ϵ

such that

$$u_\epsilon \in L^2(\Omega; L^\infty(0,T; \dot{H}^2(\mathbb{R};\mathbb{R}^3))),$$

and

$$\partial_t u_\epsilon \in L^2(\Omega; L^\infty(0,T; H^1(\mathbb{R};\mathbb{R}^3))).$$

The initial conditions

In the study of the limiting behavior of $(u_\epsilon, \partial_t u_\epsilon)$ we assume that for every $\epsilon \in (0,1)$,

$$(u_0^\epsilon, v_0^\epsilon) \in \big(\dot{H}^2(\mathbb{R};\mathbb{R}^3) \times H^1(\mathbb{R};\mathbb{R}^3)\big) \cap \mathcal{M},$$

and the following condition holds

$$\Lambda_1 := \sup_{\epsilon\in(0,1)} \big|(u_0^\epsilon, \sqrt{\epsilon} v_0^\epsilon)\big|_{\dot{H}^1(\mathbb{R})\times L^2(\mathbb{R})} < \infty,$$

and

$$\Lambda_2 := \sup_{\epsilon\in(0,1)} \sqrt{\epsilon}\big|(u_0^\epsilon, \sqrt{\epsilon} v_0^\epsilon)\big|_{\dot{H}^2(\mathbb{R})\times H^1(\mathbb{R})} < \infty.$$

Uniform estimates - Itô's formula

We fix $0 < T < R$, and for every $\epsilon > 0$ we apply the Itô formula to the mapping

$$(t, (u_\epsilon, \partial_t u_\epsilon)) \mapsto |\partial_x^{k+1} u_\epsilon|^2_{L^2(I((R-t)/\sqrt{\epsilon}))} + \epsilon\, |\partial_x^k \partial_t u_\epsilon|^2_{L^2(I((R-t)/\sqrt{\epsilon}))},$$

with $t \in [0, T]$, for $k = 0, 1$.

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with $t \in [0, T]$, for $k = 0, 1$.

Notice that we need to handle boundary terms, but we have nice cancellations and at the end we have estimates that do not have boundary terms.

Uniform estimates - Estimates in $\dot{H}^1 \times L^2$

A first consequence of the Itô formula is

$$|\partial_x u_\epsilon(t)|^2_{L^2(I((R-t)/\sqrt{\epsilon}))} + \epsilon|\partial_t u_\epsilon(t)|^2_{L^2(I((R-t)/\sqrt{\epsilon}))}$$

$$+2\gamma \int_0^t |\partial_t u_\epsilon(s)|^2_{L^2(I((R-s)/\sqrt{\epsilon}))} ds \leq |D u_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon\, |v_0^\epsilon|^2_{L^2(\mathbb{R})},$$

$\mathbb{P}$-almost surely.

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$\mathbb{P}$-almost surely.

If we take $\epsilon \in (0,1)$, this implies that for every $R > 1$

$$|\partial_x u_\epsilon(t)|^2_{L^2(-R,R)} + \epsilon|\partial_t u_\epsilon(t)|^2_{L^2(-R,R)}$$

$$+2\gamma\int_0^t |\partial_t u_\epsilon(s)|^2_{L^2(-R,R)} ds \le |Du_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon\,|v_0^\epsilon|^2_{L^2(\mathbb{R})},$$

$\mathbb{P}$-a.s. Notice that $L^2(-R,R)$ can be replaced by $L^2(\mathbb{R})$.

Uniform estimates - Estimates in $\dot{H}^2 \times H^1$

A second consequence is that for every $R > 0$ and $T > 0$

$$\mathbb{E} \sup_{t\in[0,T]} \left(|\partial_x^2 u_\epsilon(t)|^2_{L^2(-R,R)} + \epsilon |\partial_x \partial_t u_\epsilon(t)|^2_{L^2(-R,R)} \right)$$

$$+\mathbb{E} \int_0^T |\partial_x \partial_t u_\epsilon(t)|^2_{L^2(-R,R)} dt$$

$$\lesssim_{T,\Lambda_1} |D^2 u_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon |D v_0^\epsilon|^2_{L^2(\mathbb{R})} + \frac{1}{\epsilon}.$$

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$$\lesssim_{T,\Lambda_1} |D^2 u_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon |D v_0^\epsilon|^2_{L^2(\mathbb{R})} + \frac{1}{\epsilon}.$$

Notice that since the estimate above is uniform w.r.t. $R > 1$, we can replace $L^2(-R,R)$ with $L^2(\mathbb{R})$ on the left hand side.

Two fundamental facts

Thanks to the regularity of u_ϵ and $\partial_t u_\epsilon$, we can integrate by parts in $\mathbb{R}$ and obtain

$$|\partial_x u_\epsilon(t)|^2_{L^2(\mathbb{R})} + \epsilon\,|\partial_t u_\epsilon(t)|^2_{L^2(\mathbb{R})} + 2\gamma\int_0^t |\partial_t u_\epsilon(s)|^2_{L^2(\mathbb{R})}ds$$

$$= |Du_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon\,|v_0^\epsilon|^2_{L^2(\mathbb{R})}, \qquad \mathbb{P}\text{-a.s.}$$

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$$= |D u_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon\, |v_0^\epsilon|^2_{L^2(\mathbb{R})}, \qquad \mathbb{P}\text{-a.s.}$$

Another crucial fact we can prove is

$$\sup_{\epsilon \in (0,1)} \mathbb{E} \int_0^T |\partial_x^2 u_\epsilon(s)|^2_{L^2(\mathbb{R})} ds \leq c_T.$$

The heat flow equation

We introduce the deterministic heat flow equation

$$\begin{cases} \gamma_0\,\partial_t u(t,x) = \partial_x^2 u(t,x) + |\partial_x u(t,x)|^2 u(t,x), & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ u(0,x) = u_0(x), \qquad x\in\mathbb{R}, \end{cases} \tag{6}$$

where $u_0 \in M := \{u : \mathbb{R} \to \mathbb{S}^2\}$. We recall that

$$\gamma_0 = \gamma + \frac{1}{2}\,\mu(\mathbb{R}).$$

It is easy to check that $u(t) \in M$, for all $t \in \mathbb{R}^+$.

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It is easy to check that $u(t) \in M$, for all $t \in \mathbb{R}^+$.

We have proved that for every $T > 0$ and $u_0 \in \dot{H}^1(\mathbb{R}) \cap M$, problem (6) admits at most one solution in

$$L^\infty((0,T) \times \mathbb{R}) \cap L^\infty(0,T; \dot{H}^1(\mathbb{R})) \cap L^2(0,T; \dot{H}^2(\mathbb{R})).$$

A law of large numbers

We fix $u_0 \in \dot{H}^1(\mathbb{R}) \cap M$ such that

$$\lim_{\epsilon \to 0} |u_0^\epsilon - u_0|_{L^2_{\mathrm{loc}}(\mathbb{R})} = 0. \tag{7}$$

Then, for every $T > 0$, $\delta_1 < 1$ and $\delta_2 < 2$, and every $\eta > 0$, we have

$$\lim_{\epsilon \to 0} \mathbb{P}\Big(|u_\epsilon - u|_{C([0,T];H^{\delta_1}_{\mathrm{loc}}(\mathbb{R}))} + |u_\epsilon - u|_{L^2(0,T;H^{\delta_2}_{\mathrm{loc}}(\mathbb{R}))} > \eta \Big) = 0,$$

where u is the unique solution of the heat flow equation (6).

A few remarks

- Due to our assumptions, the sequence $(u_0^\epsilon - u_0)_{\epsilon\in(0,1)}$ is bounded in $H^1_{\text{loc}}(\mathbb{R})$.
  Hence (7) implies that for every $\delta < 1$

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- As a consequence of our limiting result, we obtain that we obtain that for every $T > 0$ and $u_0 \in \dot{H}^1(\mathbb{R}) \cap M$,

  the heat flow equation has a unique solution
  $u \in L^\infty(0, T; \dot{H}^1(\mathbb{R})) \cap L^2(0, T; \dot{H}^2(\mathbb{R}))$.

  This result seems to be new in the existing literature.

About the heat flow equation

In fact we prove that for every $T > 0$, $k \in \mathbb{N}$ and $u_0 \in \dot{H}^k(\mathbb{R}) \cap M$, there exists a unique solution

$$u \in L^\infty(0,T;\dot{H}^k(\mathbb{R})) \cap L^2(0,T;\dot{H}^{k+1}(\mathbb{R})),$$

with

$$\partial_t u \in L^2(0,T;H^{k-1})$$

and

$$\sup_{t\in[0,T]} |u(t)|^2_{\dot{H}^k(\mathbb{R})} + \int_0^T |u(t)|^2_{\dot{H}^{k+1}(\mathbb{R})} dt$$

$$+ \int_0^T |\partial_t u(t)|^2_{H^{k-1}(\mathbb{R})} dt \leq c_{k,T}(|u_0|_{\dot{H}^k(\mathbb{R})}).$$

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$$+\int_0^T|\partial_t u(t)|^2_{H^{k-1}(\mathbb{R})}dt\leq c_{k,T}\left(|u_0|_{\dot{H}^k(\mathbb{R})}\right).$$

Moreover,

$$|\partial_x u(t)|^2_{L^2(\mathbb{R})}+2\gamma_0\int_0^t|\partial_t u(s)|^2_{L^2(\mathbb{R})}ds=|Du_0|^2_{L^2(\mathbb{R})}.$$

A few comments about the proof

- In view of the uniform bounds we have proven for u_ϵ and $\partial_t u_\epsilon$, we can show that for every $T > 0$ and for every $\delta_1 < 1$ and $\delta_2 < 2$,

  the family of probability measures $(\mathcal{L}(u_\epsilon))_{\epsilon\in(0,1)}$ is tight in the space

$$\mathcal{X} := C([0,T]; H^{\delta_1}_{\text{loc}}(\mathbb{R})) \cap L^2(0,T; H^{\delta_2}_{\text{loc}}(\mathbb{R})).$$

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$$\mathcal{X} := C([0,T]; H^{\delta_1}_{\text{loc}}(\mathbb{R})) \cap L^2(0,T; H^{\delta_2}_{\text{loc}}(\mathbb{R})).$$

- We identify any weak limit of $(\mathcal{L}(u_\epsilon))_{\epsilon \in (0,1)}$ in $\mathcal{X}$ with the solution of the heat flow equation.

  Due to the uniqueness for the heat flow equation, we conclude that

  the whole sequence converges to u in $\mathcal{X}$ in probability.

The limiting behavior of $\partial_t u_\epsilon$ - A positive result

We have proved that

the sequence $(\partial_t u_\epsilon)_{\epsilon \in (0,1)}$ converges in probability to $\partial_t u$, with respect to the weak convergence in $L^2(0,T;L^2(\mathbb{R}))$.

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We have that for some $M > 0$

$$\{\partial_t u_\epsilon\}_{\epsilon \in (0,1)} \subset \mathcal{S}_M := \left\{\varphi \in L^2(0,T;L^2(\mathbb{R})) \ : \ |\varphi|_{L^2(0,T;L^2(\mathbb{R}))} \leq M\right\},$$

and $\partial_t u \in \mathcal{S}_M$.

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and $\partial_t u \in \mathcal{S}_M$.

Then, the metrizability of $\mathcal{S}_M$ and the convergence in probability of u_ϵ to u allow to conclude.

The limiting behavior of $\partial_t u_\epsilon$ - A negative result

We have shown that if $\sqrt{\epsilon}\,|v_0^\epsilon|_{L^2(\mathbb{R})} \to 0$, then

the sequence $(\partial_t u_\epsilon)_{\epsilon \in (0,1)}$ does not converges in probability to $\partial_t u$, with respect to the strong convergence in $L^2(0,T;L^2(\mathbb{R}))$.

The limiting behavior of $\partial_t u_\epsilon$ - A negative result

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the sequence $(\partial_t u_\epsilon)_{\epsilon \in (0,1)}$ does not converges in probability to $\partial_t u$, with respect to the strong convergence in $L^2(0,T;L^2(\mathbb{R}))$.

Recall that we proved

$$|\partial_x u_\epsilon(t)|^2_{L^2(\mathbb{R})} + \epsilon\,|\partial_t u_\epsilon(t)|^2_{L^2(\mathbb{R})} + 2\gamma \int_0^t |\partial_t u_\epsilon(s)|^2_{L^2(\mathbb{R})} ds$$

$$= |D u_0^\epsilon|^2_{L^2(\mathbb{R})} + \epsilon\,|v_0^\epsilon|^2_{L^2(\mathbb{R})}, \qquad \mathbb{P}\text{-a.s.}$$

and

$$|\partial_x u(t)|^2_{L^2(\mathbb{R})} + (2\,\gamma + \mu(\mathbb{R})) \int_0^t |\partial_t u(s)|^2_{L^2(\mathbb{R})}\, ds = |D u_0|^2_{L^2(\mathbb{R})}.$$

Therefore, we get

$$\int_0^T \int_0^t |\partial_t u_\epsilon(s)|^2_{L^2(\mathbb{R})}\, ds\, dt - \left(1 + \frac{\mu(\mathbb{R})}{2\gamma}\right) \int_0^T \int_0^t |\partial_t u(s)|^2_{L^2(\mathbb{R})}\, ds\, dt$$

$$= \frac{\epsilon\, T}{2\gamma}\, |v_0^\epsilon|^2_{L^2(\mathbb{R})} - \frac{1}{2\gamma}\Bigg(\int_0^T |\partial_x u_\epsilon(s)|^2_{L^2(\mathbb{R})}\, dt$$

$$- \int_0^T |\partial_x u(s)|^2_{L^2(\mathbb{R})}\, dt \Bigg) - \frac{\epsilon}{2\gamma} \int_0^T |\partial_t u_\epsilon(s)|^2_{L^2(\mathbb{R})}\, dt.$$

Therefore, we get

$$\int_0^T \int_0^t |\partial_t u_\epsilon(s)|^2_{L^2(\mathbb{R})}\, ds\, dt - \left(1 + \frac{\mu(\mathbb{R})}{2\gamma}\right) \int_0^T \int_0^t |\partial_t u(s)|^2_{L^2(\mathbb{R})}\, ds\, dt$$

$$= \frac{\epsilon\, T}{2\gamma}\, |v_0^\epsilon|^2_{L^2(\mathbb{R})} - \frac{1}{2\gamma}\left(\int_0^T |\partial_x u_\epsilon(s)|^2_{L^2(\mathbb{R})}\, dt \right.$$

$$\left. - \int_0^T |\partial_x u(s)|^2_{L^2(\mathbb{R})}\, dt \right) - \frac{\epsilon}{2\gamma} \int_0^T |\partial_t u_\epsilon(s)|^2_{L^2(\mathbb{R})}\, dt.$$

Since the r.h.s. converges to zero in probability, we have that the l.h.s. converges to zero in probability.

We show that if $\mu(\mathbb{R}) \neq 0$, this implies

$$\partial_t u_\epsilon \not\to \partial_t u, \quad \text{in probability} \quad \text{in } L^2(0,T;L^2(\mathbb{R})).$$

Analysis of fluctuations

Next, we want to address the nature of the stochastic fluctuations around the deterministic limit, in the case the noise has the special structure

$$w(t,x) := (\eta * w^H(t,\cdot))(x) = \int_{\mathbb{R}} \eta(x-y) w^H(t,y)\,dy,$$

where w_H is a fractional noise of Hurst index $H \in (1/2, 1)$ and η is a smooth positive kernel such that

$$1 - \mathcal{F}\eta(x) \lesssim |x|^a, \qquad x \in (-1, 1),$$

and

$$|\mathcal{F}\eta(x)| \lesssim |x|^b, \qquad x \geq 1,$$

for some constants $a \geq H - 1/2$ and $b < H - 2$.

Some notations

In what follows we define

$$y_\epsilon(t,x) := \epsilon^{H/2-1}(u_\epsilon(t,x) - u(t,x)), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Moreover, we denote by ϱ_ϵ the solution of the problem

$$\begin{cases} \gamma_0\, \partial_t \varrho_\epsilon(t,x) = \partial_x^2 \varrho_\epsilon(t,x) + |\partial_x u(t,x)|^2 \varrho_\epsilon(t,x) \\ +2(\partial_x \varrho_\epsilon(t,x) \cdot \partial_x u(t,x)) u_\epsilon(t,x) + \big(u_\epsilon(t) \times \partial_t u_\epsilon(t)\big) Q^\epsilon \partial_t w^H(t,x), \\ \varrho_\epsilon(0,x) = \epsilon^{H/2-1}(u_0^\epsilon(x) - u_0(x)), \end{cases} \tag{8}$$

where

$$Q^\epsilon h\,(x) := \frac{1}{\sqrt{\epsilon}} \int_{\mathbb{R}} \eta\Big(\frac{x-y}{\sqrt{\epsilon}}\Big) h(y)dy, \qquad h \in L^2(\mathbb{R}).$$

A *weak* central limit theorem

For every $T > 0$ and $u_0 \in \dot{H}^1(\mathbb{R}) \cap M$ such that

$$|u_0^\epsilon - u_0|_{L^2(\mathbb{R})} = o(\epsilon^{1-H/2}), \qquad 0 < \epsilon \ll 1,$$

we have

$$y_\epsilon \rightharpoonup \varrho \;\text{ in }\; L^2(\Omega; L^2(0,T; L^2(\mathbb{R}))), \;\text{ as }\; \epsilon \to 0,$$

where $\varrho \in L^2(\Omega; L^2(0,T; H))$ is the unique solution of the equation

$$\left\{ \begin{array}{l} \gamma_0\, \partial_t \varrho(t) = \partial_x^2 \varrho(t) + |\partial_x u(t)|^2 \varrho(t) + 2(\partial_x u(t) \cdot \partial_x \varrho(t)) u(t) \\[10pt] \qquad\qquad\qquad\qquad + \big(u(t) \times \partial_t u(t)\big) \partial_t w^H(t), \\[10pt] \varrho(0) = 0, \end{array} \right.$$

and u is the unique solution of the heat flow equation.

More precisely, with the notations introduced above, we have

$$\varrho_\epsilon \rightharpoonup \varrho \quad \text{in} \quad L^2(\Omega; L^2(0,T; L^2(\mathbb{R}))), \quad \text{as} \quad \epsilon \to 0,$$

and

$$\lim_{\epsilon\to 0} \mathbb{E}\left(\int_0^T |y_\epsilon(t) - \varrho_\epsilon(t)|^2_{L^2(\mathbb{R})}\, dt\right)^{1/2} = 0.$$

More precisely, with the notations introduced above, we have

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and

$$\lim_{\epsilon\to 0} \mathbb{E} \left(\int_0^T |y_\epsilon(t) - \varrho_\epsilon(t)|^2_{L^2(\mathbb{R})}\, dt \right)^{1/2} = 0.$$

Moreover, if we assume that $u_0 \in \dot{H}^3(\mathbb{R}) \cap M$, and

$$\left|(u_0^\epsilon - u_0, \sqrt{\epsilon} v_0^\epsilon)\right|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} = O(\epsilon^\beta), \qquad 0 < \epsilon \ll 1,$$

for some $\beta > 0$, then we get

$$\limsup_{\epsilon\to 0} \mathbb{E} \int_0^T |\varrho_\epsilon(t) - \varrho(t)|^2_{L^2(\mathbb{R})}\, dt \lesssim_T \mu(\mathbb{R}).$$

A few comments about the result

The analysis of fluctuations is technically demanding due to the geometry of the target manifold and the non-trivial structure of the noise which involves not only the position u, but also the velocity $\partial_t u$.

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The analysis of fluctuations is technically demanding due to the geometry of the target manifold and the non-trivial structure of the noise which involves not only the position u, but also the velocity $\partial_t u$.

The key point is showing that

proving the convergence of y_ϵ to ϱ in $L^2(0,T;L^2(\mathbb{R}))$ - whether in distribution, in probability, in mean-square, or in the weak topology of $L^2(\Omega;L^2(0,T;L^2(\mathbb{R})))$ - can be reduced to proving the analogous convergence of ϑ_ϵ to ϑ,

where ϑ_ϵ solves

$$\gamma_0\,\partial_t\vartheta_\epsilon(t) = \partial_x^2\vartheta_\epsilon(t) + (u(t)\times\partial_t u_\epsilon(t))\partial_t w_H(t), \qquad \vartheta_\epsilon(0)=0,$$

and ϑ solves

$$\gamma_0\,\partial_t\vartheta(t) = \partial_x^2\vartheta(t) + (u(t)\times\partial_t u(t))\partial_t w_H(t), \qquad \vartheta(0)=0.$$

In particular, since we can show that for every $\Phi \in L^2(\Omega; L^2(0,T; L^2(\mathbb{R})))$ it holds

$$\lim_{\epsilon \to 0} \mathbb{E} \int_0^T \langle \vartheta_\epsilon(t) - \vartheta(t), \Phi(t) \rangle_{L^2(\mathbb{R})} \, dt = 0,$$

we obtain the validity of a *weak* version of the central limit theorem, in the sense that

we show that y_ϵ converges to ϱ with respect to the weak topology of $L^2(\Omega; L^2(0,T; L^2(\mathbb{R})))$.

The *classical* CLT

Our result does not imply the CLT. The validity of a classical CLT - convergence in distribution, to what limit - remains still open and we are trying to understand that.

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The only rigorous result we can prove is

$$\limsup_{\epsilon\to 0} \mathbb{E}\left(\int_0^T |y_\epsilon(t) - \varrho(t)|^2_{L^2(\mathbb{R})}\, dt\right)^{1/2} \lesssim_T \sqrt{\mu(\mathbb{R})}.$$

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However, the fact that $\partial_t u_\epsilon$ does not converge in probability to $\partial_t u$ with respect to the strong topology of $L^2(0,T;L^2(\mathbb{R}))$, makes us believe that the CLT should give in the limit something different than ϱ.

Discussing with Francesco Caravenna this week, it seems that we could conjecture that the possible limit is something like

$$\begin{cases} \gamma_0\, \partial_t \varrho(t) = \partial_x^2 \varrho(t) + |\partial_x u(t)|^2 \varrho(t) + 2(\partial_x u(t) \cdot \partial_x \varrho(t)) u(t) \\ \quad + \big(u(t) \times \partial_t u(t)\big) \partial_t w^H(t) + \text{extra stochastic term}, \\ \varrho(0) = 0. \end{cases}$$

The extra term could involve another noise $\tilde{w}^H$ independent of w^H, with some non-linear coefficient depending on u and its derivative $\partial_t u$.

About the proof - Step 1

Recall that we have defined

$$y_\epsilon := \epsilon^{H/2-1}(u_\epsilon - u), \qquad \epsilon > 0.$$

We have proved that if $H \in [1/2, 1)$, then, for every $T > 0$ and $\alpha > 0$

$$\mathbb{E} \sup_{t \in [0,T]} |y_\epsilon(t)|^2_{L^2(\mathbb{R})} + \mathbb{E} \int_0^T |y_\epsilon(t)|^2_{H^1} dt$$

$$\lesssim_{\alpha,T} \epsilon^{-(1/2+\alpha)} + \epsilon^{H-2} |u_0^\epsilon - u_0|^2_{L^2(\mathbb{R})}, \qquad 0 < \epsilon \ll 1.$$

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$$\lesssim_{\alpha,T} \epsilon^{-(1/2+\alpha)} + \epsilon^{H-2} |u_0^\epsilon - u_0|^2_{L^2(\mathbb{R})}, \qquad 0 < \epsilon \ll 1.$$

In particular, we have

$$\mathbb{E} \sup_{t\in[0,T]} |u_\epsilon(t) - u(t)|^2_{L^2(\mathbb{R})} + \mathbb{E}\int_0^T |u_\epsilon(t) - u(t)|^2_{H^1} dt$$

$$\lesssim_{\alpha,T} \epsilon^{3/2-H-\alpha} + |u_0^\epsilon - u_0|^2_{L^2(\mathbb{R})}, \qquad 0 < \epsilon \ll 1.$$

An improved LLN

This means that if we fix u_0 and a sequence $(u_0^\epsilon)_{\epsilon \in (0,1)}$ as in the CLT, we conclude

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\epsilon(t) - u(t)|^2_{L^2(\mathbb{R})} + \mathbb{E} \int_0^T |u_\epsilon(t) - u(t)|^2_{H^1} dt = 0.$$

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Therefore, the convergence in probability of u_ϵ to u in $C([0,T]; L^2_{\text{loc}}(\mathbb{R})) \cap L^2(0,T; H^1_{\text{loc}}(\mathbb{R}))$

can be improved to mean-square convergence in $C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$.

Moreover, a bound on the rate of convergence is given, depending on the initial conditions.

About the proof - Step 2

A few slides above we introduced ϱ_ϵ, the solution of the problem

$$\begin{cases} \gamma_0\,\partial_t\varrho_\epsilon(t,x) = \partial_x^2\varrho_\epsilon(t,x) + |\partial_x u(t,x)|^2\varrho_\epsilon(t,x) \\ +2(\partial_x\varrho_\epsilon(t,x)\cdot\partial_x u(t,x))u_\epsilon(t,x) + \big(u_\epsilon(t)\times\partial_t u_\epsilon(t)\big)Q^\epsilon\partial_t w^H(t,x), \\ \varrho_\epsilon(0,x) = \epsilon^{H/2-1}(u_0^\epsilon(x) - u_0(x)), \end{cases}$$

We have proven that

$$\lim_{\epsilon\to 0}\mathbb{E}\,|y_\epsilon - \varrho_\epsilon|_{L^2(0,T;L^2(\mathbb{R}))} = 0.$$

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We have proven that

$$\lim_{\epsilon\to 0} \mathbb{E}\,|y_\epsilon - \varrho_\epsilon|_{L^2(0,T;L^2(\mathbb{R}))} = 0.$$

Notice that this limit is not trivial at all, as we have to handle several bad terms, among all $\epsilon^{H/2}\,\partial_t^2 u_\epsilon$.

About the proof - Step 3

We fix $T > 0$ and $\xi \in L^2(0,T;L^2(\mathbb{R}))$, and for every $v \in L^2(0,T;L^2(\mathbb{R}))$ we define

$$\Theta_\xi(v)(t) := \frac{1}{\gamma_0}\int_0^t e^{\frac{1}{\gamma_0}(t-s)A}\big(|\partial_x u(s)|^2 v(s)\big)ds$$

$$+\frac{2}{\gamma_0}\int_0^t e^{\frac{1}{\gamma_0}(t-s)A}\big((\partial_x u(s)\cdot\partial_x v(s))u(s)\big)ds + \xi(t).$$

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We have shown that the mapping

$$\Theta_\xi : L^2(0,T;L^2(\mathbb{R})) \to L^2(0,T;L^2(\mathbb{R}))$$

is well-defined and continuous.

Next, we have shown that for every $\xi \in L^2(0,T;L^2(\mathbb{R}))$ there is a unique $\Lambda(\xi) \in L^2(0,T;L^2(\mathbb{R}))$ such that

$$\Theta_\xi(\Lambda(\xi)) = \Lambda(\xi).$$

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$$\Theta_\xi(\Lambda(\xi)) = \Lambda(\xi).$$

Moreover, the mapping $\Lambda : L^2(0,T;L^2(\mathbb{R})) \to L^2(0,T;L^2(\mathbb{R}))$ is linear and continuous. Namely

$$|\Lambda(\xi)|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim_T |\xi|_{L^2(0,T;L^2(\mathbb{R}))}.$$

About the proof - Step 4

Now, due to our definition of ϱ_ϵ, we have that

$$\varrho_\epsilon = \Lambda(z_\epsilon),$$

where we have denoted by z_ϵ the solution of the problem

$$\gamma_0 \, \partial_t z_\epsilon(t) = \partial_x^2 z_\epsilon(t) + (u_\epsilon(t) \times \partial_t u_\epsilon(t)) Q^\epsilon dw^H(t),$$

with initial condition $z_\epsilon(0) = \epsilon^{H/2-1}(u_0^\epsilon - u_0)$.

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with initial condition $z_\epsilon(0) = \epsilon^{H/2-1}(u_0^\epsilon - u_0)$.

Thus, if we are able to prove that

$$z_\epsilon \to z, \qquad \epsilon \to 0,$$

in some appropriate sense, then

$$\varrho_\epsilon = \Lambda(z_\epsilon) \to \Lambda(z), \qquad \epsilon \to 0,$$

in some appropriate sense.

We have proved that for every $T > 0$ and $H \in (1/2, 1)$,

$$z_\epsilon \rightharpoonup z \quad \text{in} \quad L^2(\Omega; L^2(0,T; L^2(\mathbb{R}))), \qquad \text{as } \epsilon \to 0,$$

where

$$z(t) := \frac{1}{\gamma_0} \int_0^t e^{\frac{1}{\gamma_0}(t-s)A}(u(s) \times \partial_t u(s))dw^H(s), \qquad t \in [0,T].$$

We have proved that for every $T > 0$ and $H \in (1/2, 1)$,

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where

$$z(t) := \frac{1}{\gamma_0} \int_0^t e^{\frac{1}{\gamma_0}(t-s)A} (u(s) \times \partial_t u(s)) dw^H(s), \qquad t \in [0,T].$$

Therefore, we obtain

$$\varrho_\epsilon \rightharpoonup \Lambda(z) = \varrho \ \text{ in } \ L^2(\Omega; L^2(0,T; L^2(\mathbb{R}))),$$

where ϱ is the unique solution of the equation

$$\gamma_0 \partial_t \varrho(t) = \partial_x^2 \varrho(t) + |\partial_x u(t)|^2 \varrho(t) + 2(\partial_x u(t) \cdot \partial_x \varrho(t)) u(t)$$

$$+ \big(u(t) \times \partial_t u(t)\big) \partial_t w^H(t), \qquad \varrho(0) = 0.$$

One last comment

As a byproduct of our analysis, we have also shown that for $0 < \epsilon \ll 1$

$$\mathbb{E} \sup_{t\in[0,T]} |u_\epsilon(t) - u(t)|^2_{H^1}$$

$$+\mathbb{E} \int_0^T |\partial_t u_\epsilon(t) - \partial_t u(t)|^2_{L^2(\mathbb{R})} dt \lesssim_T \mu(\mathbb{R}) + \epsilon^{1\wedge 2\beta}.$$

In particular, we recover what proved in the deterministic case, under lower regularity conditions for the initial data.

Thank you

For every $u \in \dot{H}^1(\mathbb{R}) \cap M$ there exists a sequence

$$(u_n)_{n\geq 1} \subset \bigcap_{k\in\mathbb{N}} \dot{H}^k(\mathbb{R}) \cap M,$$

such that $u - u_n \in L^2(\mathbb{R})$, for every $n \in \mathbb{N}$, with

$$\lim_{n\to\infty} |u - u_n|_{H^1(\mathbb{R})} = 0.$$

In particular, this justifies condition (7).