Weak error for Numerical schemes associated with SDEs with singular drifts (works with M. Fitoussi, E. Issoglio, B. Jourdain, V. Konakov)

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Classical case

Brownian driven SDE with smooth coefficients

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,\ s \in [0, T], \qquad (SDE_{Smooth})$$

- Usual setting b, σ Lipschitz in space $\rightsquigarrow$ strong well-posedness.

(Most) Natural approximation: the Euler scheme

- Times step $h = T/N, N \in \mathbb{N},\ t_i := ih,\ \forall s \in [t_i, t_{i+1}),\ \tau_s^h := t_i$.

$$X_t^h = x + \int_0^t b(\tau_s^h, X_{\tau_s^h}^h) ds + \int_0^t \sigma(\tau_s^h, X_{\tau_s^h}^h) dW_s,\ s \in [0, T], \qquad (Euler_{Smooth}^h)$$

Strong Error: BDG-Gronwall (and some time regularity) ...

$$\mathbb{E}_x[\sup_{s \in [0,T]} |X_s - X_s^h|^p]^{\frac{1}{p}} \le C_{p,T} h^{\frac{1}{2}},\ (\mathbb{E}[|W_{t_{i+1}} - W_{t_i}|^p]^{\frac{1}{p}} = \bar{C}_p h^{\frac{1}{2}}).$$

Weak error

- Fix $T > 0$, for "some" test function φ, for X, X^h as in (SDE_{Smooth}), $(Euler^h_{Smooth})$ respectively:

$$\mathscr{E}^h_w(x, b, \sigma, T, \varphi) = \mathbb{E}_x[\varphi(X^h_T)] - \mathbb{E}_x[\varphi(X_T)].$$

⇝ Rates and assumptions:

$$\mathscr{E}^h_w(x, b, \sigma, T, \varphi) = O(h), \text{ or } \mathscr{E}^h_w(x, b, \sigma, T, \varphi) = Ch + O(h^2). \qquad (W_{Smooth})$$

Two main types of assumptions lead to that:

1. Smoothness: φ, b, σ very regular (no non-degeneracy needed), stochastic flow techniques to show **underlying Kolmogorov PDE is smooth** [TT90].
2. Non-degeneracy: b, σ smooth and (hypo)-ellipticity: "$\varphi = \delta_y$" possible in (W_{Smooth}). Huge literature (see e.g. [KM02], [BT96]...).

⇝ Rather robust approach (in particular to the noise), see [KM11] for stable driven SDEs.

Weak error analysis: ingredients

- Key role of the associated **Kolmogorov PDE**:

$$\begin{cases} \partial_s u(s,y) + \frac{1}{2}\mathrm{Tr}\big(a(s,y)D_x^2 u(s,y)\big) + b(s,y).\nabla u(s,y) = 0,\ (s,y) \in [0,T)\times\mathbb{R}^d, \\ u(T,y) = \varphi(y),\ y\in\mathbb{R}^d, \end{cases} \tag{FK}$$

$\rightsquigarrow$ Provided (FK) well posed and smooth enough:

$$\begin{aligned}
&\mathscr{E}_w^h(x,b,\sigma,T,\varphi)\\
=&\mathbb{E}[\varphi(X_T^h)] - \mathbb{E}[\varphi(X_T)] = \sum_{k=0}^{n-1}\mathbb{E}[u(t_{k+1},X_{t_{k+1}}^h) - u(t_k,X_{t_k}^h)]\\
=&\sum_{k=0}^{n-1}\mathbb{E}\Big[\int_{t_k}^{t_{k+1}} ds\Big[\Big(\partial_s + b(s,X_s^h)\cdot\nabla + \frac{1}{2}a(s,X_s^h):D^2\Big)u(s,X_s^h)\\
&+(b(t_k,X_{t_k}^h) - b(s,X_s^h))\cdot\nabla u(s,X_s^h) + \frac{1}{2}\mathrm{Tr}\big((a(t_k,X_{t_k}^h) - a(s,X_s^h))D^2u(s,X_s^h)\big)\Big]\Big]\\
=&\mathbb{E}\Big[\int_0^T ds\big(b(\tau_s^h,X_{\tau_s^h}^h) - b(s,X_s^h)\big)\cdot\nabla u(s,X_s^h)\\
&+\frac{1}{2}\mathrm{Tr}\big((a(\tau_s^h,X_{\tau_s^h}^h) - a(s,X_s^h))D^2u(s,X_s^h)\big)\Big].
\end{aligned}$$

Weak error analysis continued

$$\mathscr{E}_w^h(x,b,\sigma,T,\varphi) = \mathbb{E}[\varphi(X_T^h)] - \mathbb{E}[\varphi(X_T)]$$

$$= \mathbb{E}\Big[\int_0^T ds\big(b(\tau_s^h, X_{\tau_s^h}^h) - b(s,X_s^h)\big)\cdot\nabla u(s,X_s^h) + \frac{1}{2}\mathrm{Tr}\big((a(\tau_s^h, X_{\tau_s^h}^h) - a(s,X_s^h))D^2u(s,X_s^h)\big)\Big].$$

- If b and u smooth. Iterating Itô's formula:

  $$\mathscr{E}_w^h(x,b,\sigma,T,\varphi,h) = O(h),$$

  and (W_{Smooth}) (depending on the available smoothness).
- It $b \in C^{\gamma/2,\gamma}, a \in C^{\gamma/2,\gamma}$, $\varphi \in C^{2+\gamma}$, $\gamma \in (0,1)$, **parabolic theory** ([Fri64], [LSU68]) gives that u is smooth (gradient bounded).

  $$|\mathscr{E}_w^h(x,b,\sigma,T,\varphi)|$$
  $$\le (|\nabla u|_\infty [b]_{C^{\frac{\gamma}{2},\gamma}} + |D^2u|_\infty [a]_{C^{\frac{\gamma}{2},\gamma}})\mathbb{E}\Big[\int_0^T ds\big((s-\tau_s^h)^{\frac{\gamma}{2}} + |X_{\tau_s^h}^h - X_s^h|^\gamma\big)\Big]$$
  $$\le Ch^{\frac{\gamma}{2}},\ \ C := C(T,b,\sigma,\varphi).$$

  $\rightsquigarrow$ Mikulevicius and Platen [MP91] (see also Konakov and M. [KM17], Frikha [Fri18] for the densities).

Weak error continued: towards roughness...

Some question and remarks

- $h^{\frac{\gamma}{2}} \simeq \mathbb{E}[|W_{t_{i+1}} - W_{t_i}|^\gamma] \rightsquigarrow$ more like a strong rate...
- **Sharp rates**? It depends...
  - Non trivial diffusion coefficients: seems to naturally appear (see [KM17] and Le and Ling [LL21] in the strong case)
  - Constant diffusion case, much better bounds can be obtained.

$$\mathscr{E}_w^h(x, b, \sigma, T, \varphi) = \mathbb{E}\Big[\int_0^T ds \big(b(\tau_s^h, X_{\tau_s^h}^h) - b(s, X_s^h)\big) \cdot \nabla u(s, X_s^h)\Big].$$

- What if no smoothness of the coefficient? $\rightsquigarrow$ **exploit the regularity of the law!**

$$\begin{aligned}\mathscr{E}_w^h(x, b, \sigma, T, \varphi) =& \mathbb{E}\Big[\int_0^T ds \big(b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) - b(s, X_s^h)\big) \cdot \nabla u(s, X_s^h)\Big] \\ =& \mathbb{E}\Big[\int_0^T ds b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) \cdot (\nabla u(s, X_s^h) - \nabla u(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h))\Big] \\ &+ \int_0^T ds \int_{\mathbb{R}^d} [\mu_{\tau_s^h}^h - \mu_s^h](x, dz) b_h(s, z) \cdot \nabla u(s, z) \\ &+ \int_0^T ds \mu_s^h(x, dz)(b_h(s, z) - b(s, z)) \cdot \nabla u(s, z).\end{aligned}$$

Weak error still continued

Above:

- $U_i \overset{(\text{law})}{=} \mathcal{U}([t_i, t_{i+1}])$ $\rightsquigarrow$ Uniform independent Random variables (when no time regularity).
- b_h approximation/truncation of the drift (in the $L^q - L^p$ case or Besov setting).

Weak error continued (at last)

- Take formally $\varphi = \delta_y$, Dirac mass at fixed point y.
  One gets from previous expansion:

$$
\begin{aligned}
&\Gamma^h(0,x,T,y) - \Gamma(0,x,T,y)\\
=&\mathbb{E}\Big[\int_0^T ds b_h(U_{\lfloor s/h \rfloor}, X^h_{\tau^h_s})\cdot(\nabla_2\Gamma(s,X^h_s,T,y) - \nabla_2\Gamma(U_{\lfloor s/h \rfloor}, X^h_{\tau^h_s},T,y))\Big]\\
&+\int_0^T ds\int_{\mathbb{R}^d}[\Gamma^h(0,x,\tau^h_s,z) - \Gamma^h(0,x,s,z)]b_h(s,z).\nabla_z\Gamma(s,z,T,y)dz\\
&+\int_0^T ds\int_{\mathbb{R}^d}\Gamma^h(0,x,s,z)(b_h(s,z) - b(s,z)).\nabla_z\Gamma(s,z,T,y)dz.
\end{aligned}
$$

  - Sensitivity of the gradient of the diffusion density in the backward space variable
  - Sensitivity of the Euler scheme density in the forward time variable
  - Approximation/Truncation error for the drift
- Idea already used by Bencheikh and Jourdain [BJ20], $b \in L^\infty$.
  Total variation control $\rightsquigarrow h^{\frac{1}{2}}$. Pointwise control specified.
- Not the expansion used for the proof. Exponents are slightly worse...

SDEs handled

Model:

$$X_t = x + \int_0^t b(s, X_s)ds + Z_t, \ \ s \in [0, T], \tag{E}$$

where $(Z_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued rotationally invariant α-stable process $\alpha \in (1, 2]$.
Assumptions on the drift: β regularity parameter

- Hölder case: $b \in L^\infty([0, T], C^\beta)$, $\beta > 0$.
  Weak well-posedness holds,
  Strong if $\beta > 1 - \frac{\alpha}{2}$ see Priola [Pri12] or Chen *et al.* [CZZ21])
- Lebesgue Case $b \in L^q([0, T], L^\rho(\mathbb{R}^d))$, $\beta = 0$.

$$\frac{d}{\rho} + \frac{\alpha}{q} < \alpha - 1. \tag{KR$_\alpha$}$$

- (KR$_\alpha$) guarantees weak uniqueness (also strong if $\alpha = 2$, see [KR05]), additional conditions needed if $\alpha < 2$, see Zhang *et al.*

⇝ Critical Brownian case $\frac{d}{\rho} + \frac{2}{q} = 1$ recently addressed by Krylov (homogeneous case [Kry20]) and Röckner and Zhao (inhomogeneous case, [RZ21]).

A primer on the thresholds in the (KR_α) condition and generalization

- Intuition about (KR_α) $\rightsquigarrow$ Integrability on b needed for

$$\nabla G^\alpha b(t,x) = \int_t^T ds \nabla P^\alpha_{s-t} b(s,x) = \int_t^T ds \int_{\mathbb{R}^d} \nabla p^\alpha_{s-t}(x-y) b(s,y) dy,$$

to be bounded, where for $\alpha \in (1,2)$,

$$|\nabla p^\alpha_{s-t}(z)| \le \frac{C}{(s-t)^{\frac{1}{\alpha}+\frac{d}{\alpha}}} \frac{1}{\left(1+\frac{|z|}{(s-t)^{\frac{1}{\alpha}}}\right)^{d+\alpha}},\ p^\alpha_{s-t}(z) = g(s-t,z),\ \alpha = 2.$$

A primer on the thresholds for distributional drifts

- Besov case: $b \in L^q([0,T], B^{\beta}_{\rho,r})$, $\beta < 0$.

$$-2\beta + \frac{d}{\rho} + \frac{\alpha}{q} < \alpha - 1.$$

$\rightsquigarrow$ What is the meaning of the SDE in that case? (Virtual Solution [FIR17], Martingale Problem [DD16], [CdRM22], ...)

$\rightsquigarrow$ Continuity when $\beta = 0$ but why -2β? Paraproduct heuristics:

$$u(t,x) = \int_t^T ds \Big(P_{s-t} f(s,x) + P_{s-t}(\underbrace{b \cdot \nabla u}_{meaning?})(s,x) \Big)$$

- Set $\rho = q = \infty$, for $b \cdot \nabla u$ to exist as a distribution, condition needed:

$$\underbrace{\beta}_{regularity\ of\ the\ drift} + \underbrace{(\beta + \alpha - 1)}_{regularity\ of\ \nabla u} > 0 \iff -2\beta < \alpha - 1.$$

Besov drifts continued

- Besov case: $b \in L^q([0,T], B^{\beta}_{\rho,r})$, $\beta < 0$.

$$-2\beta + \frac{d}{\rho} + \frac{\alpha}{q} < \alpha - 1.$$

$\rightsquigarrow$ What about the dynamics? Reinforcing conditions to

$$-2\beta + \frac{2d}{\rho} + \frac{2\alpha}{q} < \alpha - 1.$$

Then (see [DD16], [CdRM22]),

$$X_t = x + \int_0^t \mathfrak{b}(s, X_s, ds) + Z_t, \qquad (\mathrm{Drift}_D)$$

where for all $(s,z) \in [0,T] \times \mathbb{R}^d$, $h > 0$,

$$\mathfrak{b}(s,z,h) := \int_s^{s+h} \int b(u,y) p_\alpha(u-s, z-y) \mathrm{d}y \mathrm{d}u = \int_s^{s+h} P^{\alpha}_{u-s} b(u,z) \mathrm{d}u, \qquad (\mathrm{Drift}_{\mathrm{loc}})$$

- Drift is a **Dirichlet process** built as a **Young integral** $\rightsquigarrow$ provides a natural approximation scheme!
- Representation could be helpful even for strong error (to be seen...)

Discretization scheme for (E): preliminary steps

$$X_t = x + \int_0^t b(s, X_s) ds + Z_t.$$

Time step: $h = T/n,\ n \in \mathbb{N},\ (U_i)_{i\geq 0}\ i.i.d.,\ U_i \overset{\text{(law)}}{=} \mathcal{U}([t_i, t_{i+1}])$.

Time randomization: no expected smoothing effect in time.

- Hölder case:

$$X_t^h = x + \int_0^t b(U_{\lfloor \frac{\tau_s^h}{h} \rfloor}, X^h_{\tau_s^h}) ds + Z_t. \qquad (Euler^h_{\text{Hölder}})$$

- Lebesgue case: **Regularity index from** (KR_α),
  - **Cut-off the drift: Scale related cut-off**:

$$b_h(t,y) = \mathbb{I}_{\{t\geq h, |b(t,y)|>0\}} \frac{|b(t,y)| \wedge (Bh^{-1+\frac{1}{\alpha}})}{|b(t,y)|} b(t,y),\ (t,y) \in [0,T] \times \mathbb{R}^d.$$

$$X_t^h = x + \int_0^t b_h(U_{\lfloor \frac{\tau_s^h}{h} \rfloor}, X^h_{\tau_s^h}) ds + Z_t. \qquad (Euler^h_{L^q - L^\rho})$$

$\rightsquigarrow$ **Allows that the drift does not dominate the magnitude of the noise**.

Schemes continued; Besov drifts

$$X_t = x + \int_0^t \mathfrak{b}(s, X_s, ds) + Z_t.$$

- Approximation scheme:

$$X^h_{t_{i+1}} = X^h_{t_i} + \mathfrak{b}(t_i, X^h_{t_i}, h) + Z_{t_{i+1}} - Z_{t_i}. \qquad (Euler^h_{Besov})$$

Define

$$\mathfrak{b}_h(s,z) := P^\alpha_{s-\tau^h_s} b(s,z) \to \mathfrak{b}(t_i, X^h_{t_i}, h) = \int_{t_i}^{t_{i+1}} \mathfrak{b}_h(s,z) ds.$$

Then, extend the dynamics of the scheme in continuous time as follows:

$$X^h_t = X^h_{\tau^h_t} + \mathfrak{b}(\tau^h_t, X^h_{\tau^h_t}, t-\tau^h_t) + Z_t - Z_{\tau^h_t} = X^h_{\tau^h_t} + \int_{\tau^h_t}^t \mathfrak{b}_h(s, X^h_{\tau^h_t}) ds + Z_t - Z_{\tau^h_t}.$$

Main Convergence result

- Let Γ, Γ^h denote the densities of the corresponding schemes.
- Let $\bar p_\alpha$ stand for an upper bound of the density of the driving noise (up to variance for $\alpha = 2$).

Theorem 3.1 (Convergence Rates)

- Hölder case (Fitoussi, M., SPA, 2025): $b \in L^\infty([0,T], C^\beta)$, $\beta > 0$

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \le Ch^{\frac{\alpha-1+\beta}{\alpha}} \bar p_\alpha(t, y-x)$$

- Lebesgue case (Jourdain, M., AAP, 2024, Fitoussi, Jourdain, M., ArXiV, 2024). *Under* (KR_α), *i.e.* $b \in L^q - L^\rho$, $\frac{\alpha}{q} + \frac{d}{\rho} < \alpha - 1$ *and* $\beta = 0$:

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \le C_c h^{\frac{\alpha-1-\frac{d}{\rho}-\frac{\alpha}{q}}{\alpha}} \bar p_\alpha(t, y-x).$$

- Besov case (Fitoussi, Issoglio, M., 2025, on ArXiV soon!). $b \in L^q([0,T], B^\beta_{\rho,r})$, $-2\beta + \frac{d}{\rho} + \frac{\alpha}{q} < \alpha - 1$, $\beta < 0$:

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \le C_c h^{\frac{\alpha-1-\frac{d}{\rho}-\frac{\alpha}{q}+2\beta}{\alpha}} \bar p_\alpha(t, y-x).$$

Some comments on the Main Convergence result

- Continuity in the convergence rate with respect to **Gap to singularity:** margin in the extended Krylov and Röckner type criteria for weak uniqueness.
- In the Besov case no need of the reinforced condition (for the limit in the Young integral, should appear for convergence of processes, we stick to marginals).

Discretization scheme: first estimates

- **Density exists for the scheme(s)**. Problem to derive **non exploding stable bounds.**

Proposition 3.2 (Density estimates for the Euler scheme)

Duhamel representation :

$$\Gamma^h(t_k, x, t, y) = p_\alpha(t - t_k, y - x) - \int_{t_k}^t \mathbb{E}\left[b_h(U_{\lfloor \frac{r}{h} \rfloor}, X^h_{\tau^h_r}) \cdot \nabla_y p_\alpha(t - r, y - X^h_r)\right] dr. \qquad (\mathrm{D}^h)$$

$\forall c > 1$, $\exists C$ not depending on $h = \frac{T}{n}$ *s.t.* $\forall k \in [\![0, n-1]\!], t \in (t_k, T], x, y, y' \in \mathbb{R}^d$,

$$C^{-1}\bar{p}_\alpha(t - t_k, y - x) \leq \Gamma^h(t_k, x, t, y) \leq C\bar{p}_\alpha(t - t_k, y - x), \qquad (\mathrm{S}^h_{HK})$$

and if $\gamma = \alpha - 1 - \left(-2\beta \mathrm{I}_{\beta<0} - \beta \mathrm{I}_{\beta \geq 0} + \frac{d}{\rho} + \frac{\alpha}{q}\right)$,

$$|\Gamma^h(t_k, x, t, y') - \Gamma^h(t_k, x, t, y)| \leq C \frac{|y - y'|^\gamma \wedge (t - t_k)^{\frac{\gamma}{\alpha}}}{(t - t_k)^{\frac{\gamma}{\alpha}}} \left(\bar{p}_\alpha(t - t_k, y - x) + \bar{p}_\alpha(t - t_k, y' - x)\right). \qquad (\mathrm{H}^h_{HK})$$

Discretization scheme for (E): yet additional estimates

- **Hölder modulus in (forward) time for the Euler scheme.**
  For all $0 \le k < \ell < n,\ t \in [t_\ell, t_{\ell+1}], x, y \in \mathbb{R}^d$,

$$|\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t_\ell, y)| \le C \frac{(t - t_\ell)^{\frac{\gamma}{\alpha}}}{(t_\ell - t_k)^{\frac{\gamma}{\alpha}}} \bar{p}_\alpha(t - t_k, y - x). \qquad (3.1)$$

Heat kernel estimates on (E) from convergence in law arguments.

- From the previous estimates letting h go to zero gives (identifying the limit):

Proposition 3.3 (Heat kernel estimates for the SDE)

Under (KR_α), *if* $(X_t)_{t\in[0,T]}$ *denote the solution to the SDE* (E),
$\forall t \in (0,T]$, X_t *admits a density* $y \to \Gamma(0,x,t,y)$ *w.r.t. the Lebesgue measure,*
$\forall c > 1$, $\exists C \in [1,\infty)$ *s.t.* $\forall t \in (0,T]$, $x, y, y' \in \mathbb{R}^d$,

$$C^{-1}\bar{p}_\alpha(t, y-x) \le \Gamma(0,x,t,y) \le C\bar{p}_\alpha(t,y-x) \qquad (\mathrm{S}_{HK})$$

and

$$|\Gamma(0,x,t,y) - \Gamma(0,x,t,y')| \le C\frac{|y-y'|^\gamma \wedge t^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}}}\Big(\bar{p}_\alpha(t,y-x) + \bar{p}_\alpha(t,y'-x)\Big). \qquad (\mathrm{H}_{HK})$$

Duhamel representation holds: $\forall t \in (0,T]$, $(x,y) \in (\mathbb{R}^d)^2$*:*

$$\Gamma(0,x,t,y) = p_\alpha(t,y-x) - \int_0^t \mathbb{E}\left[b(r,X_r)\cdot\nabla_y p_\alpha(t-r, y-X_r)\right] dr. \qquad (\mathrm{D})$$

- And even more estimates hold for this density ... (gradient w.r.t. backward variable, Hölder moduli of the gradient, similar approach than M., Pesce, Zhang [MPZ21]).

Heat kernel estimates for the diffusion Lebesgue case (Time permitting)

- Suppose $q < +\infty,\ \rho < +\infty$. Consider for $\varepsilon > 0$,

$$X_t^\varepsilon = x + Z_t - Z_s + \int_s^t b_\varepsilon(r, X_r^\varepsilon)dr,\ t \in [s, T]. \qquad (\mathrm{E}_\varepsilon)$$

  where

$$\|b_\varepsilon - b\|_{L^q - L^\rho} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

- Well known that X_t^ε admits a density $\Gamma_\varepsilon(s, x; t, \cdot)$ for $t > s$.
  $\forall c > 1,\ \exists C_\varepsilon$, s.t. $\forall (t, y) \in (s, T] \times \mathbb{R}^d$

$$|\nabla^\zeta \Gamma_\varepsilon(s, x, t, y)| \le \frac{C_\varepsilon}{(t-s)^{\frac{|\zeta|}{2}}} \bar{p}_\alpha(t-s, y-x). \qquad (\mathrm{HK}_{\varepsilon,\varepsilon})$$

- **Goal**: prove $\exists C$, independent from ε s.t. $\forall (t, y) \in (s, T] \times \mathbb{R}^d$

$$|\nabla^\zeta \Gamma_\varepsilon(s, x, t, y)| \le \frac{C}{(t-s)^{\frac{|\zeta|}{2}}} \bar{p}_\alpha(t-s, y-x). \qquad (\mathrm{HK}_\varepsilon)$$

Elements of proof for the Heat Kernel bounds

$$\begin{aligned}&\Gamma_\varepsilon(s,x,t,y)\\ =&p_\alpha(t-s,y-x)+\int_s^t du\int_{\mathbb{R}^d}\Gamma_\varepsilon(s,x,u,z)b_\varepsilon(u,z)\cdot\nabla_z p_\alpha(t-u,y-z)dz. \quad (\mathrm{D}_\varepsilon)\end{aligned}$$

We now fix $s\in[0,T)$, $c>1$ and introduce for all $u\in(s,T]$,

$$f_{\varepsilon,s}(u):=\sup_{(x,z)\in(\mathbb{R}^d)^2}\frac{\Gamma_\varepsilon(s,x,u,z)}{\bar p_\alpha(u-s,x-z)}. \quad (\mathrm{F}_\varepsilon)$$

From $(\mathrm{HK}_{\varepsilon,\varepsilon})$,

$$\forall u\in(s,T],\ f_{\varepsilon,s}(u)\le C_{\varepsilon,T,c}<+\infty.$$

$\rightsquigarrow$ allows to apply a *Gronwall type* argument.
From (D_ε), (F_ε) and usual Gaussian/stable controls:

$$\begin{aligned}&\frac{\Gamma_\varepsilon(s,x,t,y)}{\bar p_\alpha(t-s,y-x)}\le\frac{p_\alpha(t-s,y-x)}{\bar p_\alpha(t-s,y-x)}\\ &+\frac{1}{\bar p_\alpha(t-s,y-x)}\int_s^t du f_{\varepsilon,s}(u)\int_{\mathbb{R}^d}\bar p_\alpha(u-s,z-x)|b_\varepsilon(u,z)|\frac{C}{(t-u)^{\frac{1}{\alpha}}}\bar p_\alpha(t-u,y-z)dz.\end{aligned}$$

- Gaussian/stable calculus!

Elements of proof for the Heat Kernel bounds III

- **Gronwall-Volterra type Lemma** $\exists C_1 := C_1(\rho, q, d, b, T, c) < \infty$ not depending on (ε, s) s.t.

$$|\Gamma_\varepsilon(s,x,t,y)| \le C_1 \bar{p}_\alpha(t-s, y-x).$$

$\rightsquigarrow$ (HK$_\varepsilon$) holds for $\zeta = 0$!

- Estimation of the gradient. Introduce:

$$h_{\varepsilon,s}(u) := \sup_{(x,z)\in(\mathbb{R}^d)^2} \frac{|\nabla_x \Gamma_\varepsilon(s,x,u,z)|(u-s)^{\frac{1}{\alpha}}}{\bar{p}_\alpha(u-s, x-z)}. \qquad (G_\varepsilon)$$

$\rightsquigarrow$ Similar analysis up to additional singularity.

- Compactness arguments+identification of the limit give the bound.

Possible extensions

- **Multiplicative noise**: $dX_t = b(t, X_t)dt + \sigma(t, X_t)dZ_t$.
  - $\sigma(t, x) = \sigma(x)$ and sigma u.e., bounded and Lipschitz, should work for homogeneity reasons (see also Zhang [Zha11] in for the continuous case: $W^{1,p}$ should work as well).
  - difficulty to handle mere measurability in time (appears in density transitions).

Related Results

- Strong Error (K. Lê and C. Ling, [LL21]): under (KR_α), $\alpha = 2$.

$$\mathbb{E}[\sup_{t\in[0,T]} |X_t - \tilde{X}_t^h|^p]^{\frac{1}{p}} \le C_p h^{\frac{1}{2}} |\ln(h)|.$$

⇝ Stochastic sewing arguments

⇝ Pathwise analysis (no gradient involved)

- Weak error with test functions

⇝ In connection with weak error, Z. Hao, K. Le, C. Ling (ArXiV, 2024), in the $L^\infty - L^p$ setting for tamed Euler scheme (i.e. similar than Besov case) obtain weak rate of order $h^{\frac{1}{2}}$ for a bounded test function (important).

- Links between weak and strong, benefit from better smoothness estimates on test functions.

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