Fokker–Planck equations with integral drift and McKean–Vlasov SDEs

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Introduction

$$\begin{aligned}&\frac{\partial u}{\partial t}-\Delta\beta(u(t,x))+\operatorname{div}((D(x)b(u(t,x))\\&\qquad+(K*u(t,\cdot))(x)u(t,x))=0\text{ in }(0,\infty)\times\mathbb{R}^d,\\&u(0,x)=u_0(x),\ x\in\mathbb{R}^d,\end{aligned}\tag{1}$$

$d\geq 2$, $\beta:\mathbb{R}\to\mathbb{R}$, $b:\mathbb{R}\to\mathbb{R}^+$, $D:\mathbb{R}^d\to\mathbb{R}^d$, $K:\mathbb{R}^d\to\mathbb{R}^d$.

(i) $\beta\in C^1(\mathbb{R})$, $\beta(0)=0$, $\beta'(r)\geq\alpha>0$, $\forall r\in\mathbb{R}$.

(ii) $D\in L^\infty(\mathbb{R}^d;\mathbb{R}^d)\cap L^2(\mathbb{R}^d;\mathbb{R}^d)$, $(\operatorname{div} D)^-=0$.

(iii) $b\in C^1(\mathbb{R})$, $b(r)\geq 0$, $\forall r\in\mathbb{R}$.

(iv) $K\in C^1(\mathbb{R}^d\setminus 0;\mathbb{R}^d)\cap L^1(B_1;\mathbb{R}^d)\cap L^p(B_1^c;\mathbb{R}^d)$, where $p\in[1,2]$,

$$(\operatorname{div} K)^-\in L^\infty(\mathbb{R}^d),\ (K(x)\cdot x)^-|x|^{-1}\in L^\infty(B_1),\tag{2}$$

where $B_r=\{x\in\mathbb{R}^d;\ |x|\leq r\}$, $B_r^c=\{x\in\mathbb{R}^d;\ |x|>r\}$, $\forall r>0$.

NFPE (1) is related to the McKean–Vlasov SDE

$$\begin{aligned} dX_t &= (D(X_t)b(u(t,X_t)) + (K * u(t,\cdot))(X_t))dt \\ &\quad + \left(\frac{2\beta(u(t,X_t))}{u(t,X_t)}\right)^{\frac{1}{2}} dW_t,\ t \geq 0, \end{aligned} \tag{3}$$

through the stochastic representation

$$\mathcal{L}_{X_t}(x) = u(t,x),\ \forall t \geq 0;\ u_0(x) = \mathbb{P} \circ X_0^{-1},\ x \in \mathbb{R}^d. \tag{4}$$

Here, $\mathcal{L}_{X_t}$ is the density of the marginal law $\mathbb{P} \circ X_t^{-1}$ of X_t with respect to the Lebesgue measure.

The special case $K \equiv 0$ was studied in the works [2]–[5].

1°. **Riesz type kernels.** The kernel

$$K(x) = \mu x|x|^{s-d-2},\ 0 < s < d,\ \mu > 0,\ d > 2, \tag{5}$$

is derived from the Riesz potential, $K(x) \equiv \nabla I_s(x)$, where

$$\begin{aligned} I_s(x) &= -\frac{\mu}{d-s}\,|x|^{s-d} && \text{if } 0 < s < d,\ d > 2 \\ I_s(x) &= \mu \log(x) && \text{if } d = 2. \end{aligned}$$

Condition (iv) holds if $2 < s < d + 2$.
We note that, for $0 < s < d$ and

$$\mu = (d-s)\pi^{\frac{d}{2}}\, 2^s\, \Gamma\left(\frac{s}{2}\right) \Big/ \Gamma\left(\frac{d-s}{2}\right),$$

we have

$$(-\Delta)^{-\frac{s}{2}} f = I_s * f,\ \forall f \in L^1 \cap L^2. \tag{6}$$

2°. Bessel kernels

$$K(x) = \nabla G_\alpha(x),\ x \in \mathbb{R}^d \setminus \{0\},\ 0 < \alpha < d, \tag{7}$$

where G_α is the Bessel potential defined by

$$\mathcal{F}(G_\alpha)(\xi) = -(1 + 4\pi^2|\xi|)^{-\frac{\alpha}{2}},\ \xi \in \mathbb{R}^d.$$

(Here, $\mathcal{F}(G_\alpha)$ is the Fourier transform of G_α.) In analogy with (6), we have

$$(I - \Delta)^{-\frac{\alpha}{2}} f = G_\alpha * f,\ \nabla((I - \Delta)^{-\frac{\alpha}{2}} f) = K * f, \tag{8}$$

$$\begin{aligned} G_\alpha(x) &= -H_{\frac{\alpha-d}{2}}(|x|)|x|^{\frac{\alpha-d}{2}},\ 0 < \alpha < d+1,\ x \in \mathbb{R}^d, \\ H_\nu(r) &\equiv \mu_\nu\, r^\nu\, e^{-\nu} \int_0^\infty e^{-tr}\, t^{\nu-\frac{1}{2}} \left(1 + \tfrac{1}{2}\, t\right)^{\nu-\frac{1}{2}} dt,\ \nu > \tfrac{1}{2},\ r > 0. \end{aligned}$$

Such an example is the Keller–Segal model [11]

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta\beta(u) + \operatorname{div}(u\nabla v) &= 0 \quad \text{in } (0,\infty)\times\mathbb{R}, \\ (-\Delta)^{\alpha} v &= u \quad \text{in } (0,\infty)\times\mathbb{R}^d, \\ (I-\Delta)^{\alpha} v &= u \quad \text{where } 0<\alpha\le 1, \end{aligned} \tag{9}$$

of the chemotaxis dynamics of biological populations in presence of an anomalous diffusion.

If $K = \nabla W,\ D = -\nabla\Phi,\ \Phi : \mathbb{R}^d \to \mathbb{R}$, then (9) is associated with the entropy functional

$$\begin{aligned} E(u) = &\int_{\mathbb{R}^d} \left(\int_1^{u(x)} \frac{\beta'(\tau)}{b(\tau)\tau}\, d\tau + \Phi(x)u(x) \right) dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)u(x)u(y)dxdy. \end{aligned}$$

The Biot–Savart kernel

$$K(x) = \frac{1}{\pi}\, x^{\perp}|x|^{-2}$$

satisfies (2), but not (iv).

(A direct treatment in V.B., M. Röckner, D. Zhang [6].)

The existence and uniqueness of the solution $u : [0,\infty) \to H^{-1}(\mathbb{R}^{+})$ to NFPE (1) in the set $\mathcal{P}$ of probability densities,

$$\mathcal{P} = \left\{ u \in L^1(\mathbb{R}^d);\ u \geq 0,\ \text{a.e. in } \mathbb{R}^d, \int_{\mathbb{R}^d} u(x)dx = 1 \right\}. \tag{10}$$

Moreover, for $u_0 \in \mathcal{P} \cap L^{\infty}(\mathbb{R}^d)$, the flow $t \to u(t) = S(t)u_0$ is a continuous semigroup of quasi-contractions in the Sobolev space $H^{-1}(\mathbb{R}^d)$ which is everywhere differentiable from the right on $[0,\infty)$ (see Theorem 1).

Notations and definitions

Denote by H^1 the Sobolev space

$$H^1(\mathbb{R}^d) = \left\{ u \in L^2;\ \frac{\partial u}{\partial x_i} \in L^2,\ i = 1, ..., d \right\}$$

and by H^{-1} the dual of H^1. The space H^{-1} is endowed with the scalar product

$$\langle u_1, u_2 \rangle_{-1} = ((I - \Delta)^{-1} u_1, u_2)_2,$$

where $(\cdot,\cdot)_2$ is the scalar product of the space L^2. We denote by $|\cdot|_{-1}$ the corresponding Hilbertian norm of H^{-1}, that is,

$$|u|_{-1} = ((I - \Delta)^{-1} u, u)_2^{\frac{1}{2}},\ \forall u \in H^{-1}.$$

For $0 < T \le 0$, we shall denote by $C([0,T]; H^{-1}$ the space of all H^{-1}-valued continuous functions on $[0,T]$. $W^{1,p}([0,T]; H^{-1})$, $1 \le p \le \infty$, is the space $\left\{ u \in L^\infty(0,T;H^{-1});\ \frac{du}{dt} \in L^p(0,T;H^{-1}) \right\}$.

The nonlinear operator $A : D(A) \subset H^{-1} \to H^{-1}$ is said to be *quasi-m-accretive* if there is $\omega > 0$ such that, for all $\lambda \in \left(0, \frac{1}{\omega}\right)$,

$$\left|(I+\lambda A)^{-1}f_1 - (I+\lambda A)^{-1}f_2\right|_{-1} \le (1-\lambda\omega)^{-1}|f_1 - f_2|_{-1}, \quad \forall f_1, f_2 \in H^{-1}. \tag{11}$$

Equivalently,

$$\begin{aligned} &\langle Au_1 - Au_2, u_1 - u_2\rangle_{-1} \ge -\omega|u_1 - u_2|_{-1}, \ \forall u_1, u_2 \in D(A), \\ &R(I+\lambda A) = H^{-1}, \ \forall \lambda \in (0, \lambda_0), \end{aligned} \tag{12}$$

for some $\lambda_0 \in \left(0, \frac{1}{\omega}\right)$.

If A is *quasi-m-accretive*, then it generates on $\overline{D(A)}$ (the closure of $DF(A)$) a continuous semigroup of ω-quasi-contractions $S(t)$, that is

$$\begin{aligned} &\frac{d^+}{dt} S(t)u_0 + AS(t)u_0 = 0, \ \forall t > 0, \\ &\lim_{t\to 0} S(t)u_0 = u_0, \ S(t+s)u_0 = S(t)S(s)u_0, \ \forall t, s \ge 0, \end{aligned} \tag{13}$$

for all $u_0 \in D(A)$.

$$S(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0 \quad \text{in } H^{-1} \tag{14}$$

uniformly on compact intervals $[0, T] \subset [-, \infty)$ and

$$|S(t)u_0 - S(t)v_0|_{-1} \le \exp(\omega t)|u_0 - v_0|_{-1},\ \forall u_0, v_0 \in \overline{D(A)},\ t \ge 0. \tag{15}$$

$u(t) = S(t)u_0$ is for $u_0 \in D(A)$ a *smooth* (*differentiable*) *solution* (in the space H^{-1}) to the Cauchy problem

$$\frac{du}{dt} + Au = 0,\ t \ge 0;\ u(0) = u_0. \tag{16}$$

Weak solution to (1)

$$u \in C([0,\infty);H^{-1}),\ \frac{du}{dt} \in L^2(0,T;H^{-1}),\ \forall T>0,$$

$$\beta(u) \in L^2(0,T;H^1),\ Db(u)+(K*u)u \in L^2(0,T;L^2),\ \forall T>0,$$

$$\begin{aligned} {}_{H^{-1}}\left(\frac{du}{dt}(t),\varphi\right)_{H^1} &+ \int_{\mathbb{R}^d} (\nabla\beta(u(t,x)) - (Db(u(t,x))u(t,x) \\ &+K(u(t,x))*u(t))u(t,x))\cdot\nabla\varphi(x))dx = 0,\ \forall\varphi\in H^1, \\ &\text{a.e. } t>0, \end{aligned}$$

$$u(0,x)=u_0(x),\ x\in\mathbb{R}^d.$$

Theorem 1

Let $u_0 \in \mathcal{P} \cap L^\infty$*. Then, there is a unique weak solution* $u = u(t, u_0)$ *such that*

$$u(t) \in \mathcal{P},\ \forall t \geq 0,\ u \in L^\infty((0,T) \times \mathbb{R}^d),\ \forall T > 0.$$

Moreover,

$$|u(t,u_0) - u(t,u_0)|_{-1} \leq \exp(\omega t)|u_0 - v_0|_{-1},\ \forall t > 0,\ u_0, v_0 \in \mathcal{P} \cap L^\infty,$$

$$0 \leq u(t,x) \leq \exp(\gamma t)|u_0|_\infty,\ \forall (t,x) \in (0,\infty) \times \mathbb{R}^d.$$

$$t \to u(t,u_0) \textit{ is narrowly continuous}.$$

Let

$$\mathcal{K} = \overline{\mathcal{P} \cap L^\infty},\ S(t)u_0 = u(t, u_0).$$

Then $S(t) : \mathcal{K} \to \mathcal{K}$ is a continuous semigroup of quasi-contractions on $\mathcal{K}$ and by Kōmura's theorem it is generated by a quasi-m-accretive operator A_0, on $\mathcal{K}$

$$\frac{d^+}{dt} S(t)u_0 + A_0 S(t)u_0 = 0,\ \forall t \geq 0,\ \forall u_0 \in D(A_0),$$

where $D(A_0)$ is a dense subset of $\mathcal{K}$.

Conclusion. $t \to u(t, u_0)$ is *differentiable on the right on a dense subset of* $\mathcal{K} = H^{-1} \cap \mathcal{P}_0$ *where* $\mathcal{P}_0$ *is the set of all probability measures.*

Proof of Theorem 1

We set, for $\varepsilon > 0$,

$$\beta_\varepsilon(r) \equiv \varepsilon r + \beta((1+\varepsilon\beta)^{-1}r), \ \forall r \in \mathbb{R},$$

$$\varphi_\varepsilon(r) = \frac{1}{\varepsilon}\,\frac{r}{|r|} \text{ if } |r| \geq \frac{1}{\varepsilon}, \ \varphi_\varepsilon(r) = r \text{ if } |r| \leq \frac{1}{\varepsilon},$$

$$K_\varepsilon(x) \equiv \eta\left(\frac{|x|}{\varepsilon}\right)K(x), \ b_\varepsilon(r) \equiv (1-\eta(\varepsilon r))b(r),$$

where $\eta \in C^1([0,\infty))$ is such that $0 \leq \eta'(r) \leq 1$, $0 \leq \eta(r) \leq 1$, $\forall r \geq 0$, and

$$\eta(r) = 0, \ \forall r \in [0,1]; \ \eta(r) = 1, \ \forall r \geq 2.$$

We also set $j(r) \equiv \int_0^r \beta(s)ds$ and $j_\varepsilon(r) \equiv \int_0^t \beta_\varepsilon(s)$.

Now, we define the operator $A_\varepsilon : D(A_\varepsilon) \subset H^{-1} \to H^{-1}$,

$$A_\varepsilon(u) = -\Delta\beta_\varepsilon(u) + \operatorname{div}((Db_\varepsilon(u)u + (K_\varepsilon * \varphi_\varepsilon(u))\varphi_\varepsilon(u)), \quad \forall u \in D(A_\varepsilon) = H^1. \tag{17}$$

Lemma 2

A_ε is quasi-m-accretive in H^{-1} and for $\lambda \in (0, \lambda_\varepsilon)$

$$(I + \lambda A_\varepsilon)^{-1} f \geq 0, \text{ a.e. in } \mathbb{R}^d \text{ if } f \geq 0, \tag{18}$$

$$(I + \lambda A_\varepsilon)^{-1} (\mathcal{P} \cap H^{-1}) \subset \mathcal{P}, \tag{19}$$

Moreover, we have

$$0 \leq (I + \lambda A_\varepsilon)^{-1} f \leq N, \ \text{ a.e. in } \mathbb{R}^d, \tag{20}$$

for all $N > 0$ and $f \in \mathcal{P} \cap L^\infty$ such that, for $0 < \lambda < \frac{1}{\gamma}$,

$$0 \leq f \leq (1 - \lambda\gamma) N, \text{ a.e. in } \mathbb{R}^d, \tag{21}$$

$$\gamma = (\text{div } K)^- |_\infty + \left\| \frac{(K(x) \cdot x)^-}{|x|} \right\|_{L^\infty(B_2)}. \tag{22}$$

Proof. We set $b_\varepsilon^*(r) = b_\varepsilon(r) r$, $b^*(r) = b(r) r$, $\forall r \in \mathbb{R}$, and note that

$$b_\varepsilon^*, (b_\varepsilon^*)' \in L^\infty(\mathbb{R}), \ \forall \varepsilon > 0. \tag{23}$$

Lemma 3

For each $u_0 \in H^{-1}$ *there is a unique mild solution* $u_\varepsilon \in C([0,\infty); H^{-1})$ *to the Cauchy problem*

$$\frac{du_\varepsilon}{dt} + A_\varepsilon u_\varepsilon = 0 \text{ on } (0,\infty), \qquad (24)$$
$$u_0(0) = u_0,$$

which is a smooth solution, that is, $\frac{d^+}{dt}u_\varepsilon(t)$ *exists everywhere on* $[0,\infty)$ *if* $u_0 \in H^1 = D(A_\varepsilon)$. *Moreover, if* $u_0 \in \mathcal{P} \cap H^{-1}$, *then* $u_\varepsilon(t) \in \mathcal{P}$, $\forall t \geq 0$. *If* $j(u_0) \in L^2$, *then, for every* $T > 0$,

$$u_\varepsilon, \beta_\varepsilon(u_\varepsilon) \in L^2(0,T;H^1) \qquad (25)$$

$$\frac{du_\varepsilon}{dt} \in L^2(0,T;H^{-1}) \qquad (26)$$

$$\mathrm{div}((K_\varepsilon * u_\varepsilon) \in L^2(0,T;L^2) \qquad (27)$$

$$\frac{du_\varepsilon}{dt}(t) - \Delta\beta_\varepsilon(u_\varepsilon(t)) + \mathrm{div}(Db_\varepsilon(u_\varepsilon(t))u_\varepsilon(t) + (K_\varepsilon * \varphi_\varepsilon(u_\varepsilon))u_\varepsilon)(t)) = 0,\ a.e.\ t > 0, \qquad (28)$$

$$0 \leq u_\varepsilon(t,x) \leq \exp(\gamma t)|u_0|_\infty,\ \ a.e.\ (t,x) \in (0,\infty) \times \mathbb{R}^d. \qquad (29)$$

Hence, on a subsequence $\{\varepsilon\} \to 0$, we have

$$
\begin{array}{rcll}
u_\varepsilon & \to & u^* & \text{weakly in } L^2_{\text{loc}}(0,\infty;H^1) \\
\dfrac{du_\varepsilon}{dt} & \to & \dfrac{du^*}{dt} & \text{weakly in } L^2_{\text{loc}}(0,\infty;H^{-1}) \\
u_\varepsilon & \to & u^* & \text{weak-star in } L^\infty((0,\infty)\times\mathbb{R}^d), \\
 & & & \text{weakly in } L^2_{\text{loc}}(0,\infty), \text{ and} \\
 & & & \text{strongly in } L^2_{\text{loc}}(0,\infty;L^2) \\
Db_\varepsilon(u_\varepsilon)u_\varepsilon & \to & Db(u^*)u^* & \text{weakly in } L^1_{\text{loc}}(0,\infty;L^2) \\
\beta_\varepsilon(u_\varepsilon) & \to & \beta(u^*) & \text{weakly in } L^2_{\text{loc}}(0,\infty;H^1) \\
(K_\varepsilon * \varphi_\varepsilon(u_\varepsilon))\varphi_\varepsilon(u_\varepsilon) & \to & v^* & \text{weak-star in } L^\infty((0,\infty)\times\mathbb{R}^d).
\end{array}
\qquad (30)
$$

Clearly, u^* satisfies the equation

$$\frac{du^*}{dt} - \Delta\beta(u^*) + \mathrm{div}(v^* + Db(u^*)u^*) = 0 \text{ in } \mathcal{D}'(0,\infty)\times\mathbb{R}^d$$
$$u^*(0) = u_0.$$

Since

$$|K_\varepsilon * \varphi_\varepsilon(u_\varepsilon)|_\infty \le C,\ \forall \varepsilon > 0,$$

and by (30),

$$K_\varepsilon * \varphi_\varepsilon(u_\varepsilon)\varphi_\varepsilon(u_\varepsilon) = (K_\varepsilon * u_\varepsilon)u_\varepsilon \to (K * u^*)u^*,$$

a.e. in $(0,\infty)\times\mathbb{R}^d$ as $\varepsilon\to 0$, we infer that $v^* \equiv (K * u^*)u^*$. We also have

$$0 \le \left(I + \frac{t}{n}A\right)^{-n} u_0 \le \left(1 - \frac{t}{n}\gamma\right)^{-n} |u_0|_\infty,\ \forall n\in\mathbb{N}.$$

Hence,

$$0 \le u_\varepsilon(t) \le \exp(\gamma t)|u_0|_\infty,\ \forall\varepsilon > 0.$$

More precisely, we have

$$\frac{du^*}{dt} - \Delta\beta(u^*) + \operatorname{div}(Db(u^*)u^* + (K * u^*)u^*) = 0 \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d) \tag{31}$$
$$u^*(0,x) = u_0(x),\ x \in \mathbb{R}^d$$

$$u^*, \beta(u^*) \in L^2_{\text{loc}}(0,\infty; H^1), (K * u^*)u^* \in L^\infty((0,\infty)\times\mathbb{R}^d) \tag{32}$$

$$u^*(t) \in \mathcal{H}_N,\ \forall t \geq 0 \tag{33}$$

$$\frac{du^*}{dt} \in L^2_{\text{loc}}(0,\infty; H^{-1}) \tag{34}$$

It also follows

$$|u^*(t,u_0) - u^*(t,v_0)|_{-1} \leq \exp(\gamma_0 t)|u_0 - v_0|_{-1},\ \forall u_0, v_0 \in \mathcal{P}\cap L^\infty.$$

We set

$$S(t)u_0 = u^*(t),\ t \geq 0,\ u_0 \in \mathcal{P}\cap L^\infty, \tag{35}$$

and note that $S(t)(\mathcal{P}\cap L^\infty) \subset \mathcal{P}\cap L^\infty$, $\forall t \geq 0$, the function $t \to S(t)u_0$ is continuous (in H^{-1}) on $[0,\infty)$ and

$$S(t+s)u_0 = S(t)S(s)u_0,\ \forall t,s \geq 0,\ u_0 \in \mathcal{P}\cap L^\infty,$$
$$|S(t)u_0 - S(t)v_0|_{-1} \leq \exp(\gamma_0 t)|u_0 - v_0|_{-1},\ \forall t \geq 0,\ u_0, v_0 \in \mathcal{P}\cap L^\infty.$$

In other words, $S(t)$ *is a continuous semigroup of* γ_0*-contractions on* $\mathcal{P}\cap L^\infty \subset H^{-1}$ which extends by continuity on $\mathcal{K}=\overline{\mathcal{P}\cap L^\infty}$. Then, by Kōmura's theorem (see [1], p. 175) there is a quasi-m-accretive operator A_0 with the domain $D(A_0)$, which generates the semigroup $S(t)$ on $\overline{D(A_0)}=\mathcal{K}$, that is,

$$\frac{d^+}{dt}\,S(t)u_0+A_0S(t)u_0=0,\ \forall t\ge 0,\ u_0\in D(A_0), \tag{36}$$

and the function $t\to \frac{d}{dt}\,S(t)$ exists and is everywhere continuous on $(0,\infty)$ except a countable set of t,

$$A_0u_0=-\Delta\beta(u_0)+\mathrm{div}(Db(u_0)u_0)+(K*u_0)u_0),\ \forall u_0\in D(A_0).$$

Long time behaviour (H-theorem)

Let $K = -\nabla W,\ W(x) \equiv W(-x)$,

$$E(u) = \int_{\mathbb{R}^d} \left(\int_1^{u(x)} \frac{\beta'(\tau)}{b(\tau)\tau}\, d\tau + \Phi(x) \right) dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y)u(x)u(y)dxdy$$

(The entropy functional)

Theorem 4

One has, for $b \equiv 1$ and $u \to W * u$, a positive operator

$$E(S(t)u_0) + \int_s^t \Psi(S(\tau)u_0)d\tau \le E(S(s)u_0),\ \forall u_0 \in \mathcal{P} \cap L^\infty,\ 0 \le s \le t < \infty,$$

$$\Psi(u) \equiv \int_{\mathbb{R}^d} u \left| \frac{\nabla \beta(u)}{u} - D - K * u \right|^2 dx,$$

$\{S(t)u_0;\ t \ge 0\}$ *is compact in* $L^1_{\rm loc}(\mathbb{R}^d)$ *and any* $u_0 = \lim\limits_{t_n \to \infty} S(t_n)u_0$ *in* $L^1_{\rm loc}$ *is an equilibrium point of the system,*

$$\nabla \beta(u_\infty) - Du_\infty - (K * u_\infty)u_\infty \equiv 0.$$

The uniqueness of distributional solutions

We shall assume here that *besides* (i)–(iv) *the following condition hold*

(v) $K \in L^\infty(B_1^c; \mathbb{R}^d)$.

We recall that the function $u \in L^1_{\text{loc}}([0,T) \times \mathbb{R}^d)$ is said to be a *distributional solution* to NFPE (1) on $Q_T = (0,T) \times \mathbb{R}^d$ if

$$\beta(u) \in L^1_{\text{loc}}(Q_T) \tag{37}$$

$$(Db(u) + K * u)u \in L^1_{\text{loc}}(Q_T; \mathbb{R}^d) \tag{38}$$

$$\begin{aligned} \int_{Q_T} (u(t,x)\frac{\partial \varphi}{\partial t}(t,x) + \beta(u(t,x))\Delta\varphi(t,x) + u(t,x)(D(x)b(u(t,x)) \\ +(K * u(t))(x)) \cdot \nabla\varphi(t,x))dtdx + \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx = 0, \\ \forall \varphi \in C_0^\infty([0,T) \times \mathbb{R}^d). \end{aligned} \tag{39}$$

Theorem 5

Let $u_1, u_2 \in L^1(Q_T) \cap L^\infty(Q_T)$ be two distributional solutions to (1) such that

$$\lim_{t\downarrow 0} \text{ess sup} \int_{\mathbb{R}^d} (u_1(t,x) - u_2(t,x))\psi(x)dx = 0,\ \forall \psi \in C_0^\infty(\mathbb{R}^d). \tag{40}$$

Then $u_1 \equiv u_2$.

Proof. We set $z = u_1 - u_2$, $w = \beta(u_1) - \beta(u_2)$ and get for z the equation

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta w + \text{div}(D((b(u_1) - b(u_2))u_1 + b(u_2)z)) \\ &\quad + \text{div}((K * z)u_1 - z(K * u_2)) = 0 \text{ in } \mathcal{D}'(Q_T). \end{aligned} \tag{41}$$

Consider the operator $\Phi_\varepsilon = (\varepsilon I - \Delta)^{-1} \in L(L^2, L^2)$. We have

$$\varepsilon\Phi_\varepsilon(y) - \Delta\Phi_\varepsilon(y) = y \text{ in } \mathcal{D}'(\mathbb{R}^d),\ \forall y \in L^2. \tag{42}$$

We see that $\Phi_\varepsilon \in L(L^2, H^2)$ and applying Φ_ε to (41) we get

$$\begin{aligned} &\frac{d}{dt}\Phi_\varepsilon(z(t)) + w(t) - \varepsilon\Phi_\varepsilon(w(t)) \\ &\quad +\mathrm{div}(\Phi_\varepsilon(z(t))(D((b(u_1(t)) - b(u_2(t))) + b(u_2(t))z(t))) \\ &\quad +\mathrm{div}(\Phi_\varepsilon((K * z(t))u_1(t) - z(t)(K * u_2(t)))) = 0, \text{ a.e. } t \in (0,T). \end{aligned} \tag{43}$$

$$h_\varepsilon(t) = (\Phi_\varepsilon(z(t)), z(t))_2 = \varepsilon|\Phi_\varepsilon(z(t))|_2^2 + |\nabla\Phi_\varepsilon(z(t))|_2^2,\ t \in [0,T], \tag{44}$$

$$\begin{aligned} h'_\varepsilon(t) = 2\left(\frac{d}{dt}\Phi_\varepsilon(z(t)), z(t)\right)_2 &= -2(w(t), z(t))_2 + 2\varepsilon(\Phi_\varepsilon(w(t)), z(t))_2 \\ +((K*z(t))u_1(t) - z(t)(K*u_2(t)), \nabla\Phi_\varepsilon(z(t)))_2 &+ (\Psi(t), \nabla\Phi_\varepsilon(z(t)))_2, \\ &\text{a.e. } t \in (0,T), \end{aligned} \tag{45}$$

$$\Psi \equiv D(b(u_1) - b(u_2)u_1 + b(u_2)z)). \tag{46}$$

This yields

$$h'_\varepsilon(t)+2\alpha|z(t)|_2^2\le 2\varepsilon(\Phi_\varepsilon(w(t)),z(t))_2+C|z(t)|_2|\nabla\Phi_\varepsilon(z(t))|_2, \quad \text{a.e. } t\in(0,T), \tag{47}$$

and so, by (44) we get

$$h'_\varepsilon(t)+\alpha|z(t)|_2^2\le Ch_\varepsilon(t)+2\varepsilon(\Phi_\varepsilon(w(t)),z(t))_2,\ \text{a.e. } t\in(0,T).$$

We have

$$|w(t)|\le C|z(t)|_2,\ \forall t\in(0,T). \tag{48}$$

$$\sqrt{\varepsilon}|\Phi_\varepsilon(w(t)),z(t))_2|\le C,\ \text{a.e } t\in(0,T). \tag{49}$$

It follows that

$$h_\varepsilon(t) \leq C \int_0^t h_\varepsilon(s)ds + C\sqrt{\varepsilon},\ \forall t \in [0,T].$$

Hence,

$$h_\varepsilon(t) \leq C\sqrt{\varepsilon}\exp(Ct),\ \forall t \in [0,T],$$

and so $\lim\limits_{\varepsilon\to 0} h_\varepsilon(t) = 0$ uniformly on $[0,T]$.

Recalling (44), we get for $\varepsilon \to 0$

$$\begin{array}{rcl} \nabla\Phi_\varepsilon(z) & \to & 0 \quad \text{in } L^2(0,T;(L^2)^d) \\ \sqrt{\varepsilon}\,\Phi_\varepsilon(z) & \to & 0 \quad \text{in } L^2(0,T;L^2) \\ \Delta\Phi_\varepsilon(z) & \to & 0 \quad \text{in } L^2(0,T;H^{-1}) \end{array}$$

Taking into account that $\varepsilon\Phi_\varepsilon(z) - \Delta\Phi_\varepsilon(z) = z$, we infer that $z = 0$ on $(0,T) \times \mathbb{R}^d$, as claimed. □

Now, we shall prove for later use a similar uniqueness result for the distributional solution v to the "freezed" linear version of NFPE (39), that is,

$$\frac{\partial v}{\partial t} - \Delta\left(\frac{\beta(u)}{u}\,v\right) + \operatorname{div}((Db(u) + K * u)v) = 0 \text{ in } \mathcal{D}'(0,T) \times \mathbb{R}^d,$$
$$v(0) = v_0, \tag{50}$$

where u is the solution to (1) given by Theorem 1. In other words,

$$\int_{Q_T} \left(\frac{\partial \varphi}{\partial t} + \frac{\beta(u)}{u}\,\Delta\varphi + (Db(u) + K * u)\cdot\nabla\varphi\right) dtdx$$
$$+ \int_{\mathbb{R}^d} v_0(z)\varphi(0,x)dx,\ \forall \varphi \in C_0^\infty([0,T) \times \mathbb{R}^d), \tag{51}$$

Theorem 6

Let $u \in L^\infty(Q_T) \cap L^1(Q_T)$ *be the solution to* (1) *and let*

$$v_1, v_2 \in L^1(Q_T) \cap L^\infty(Q_T)$$

be two distributional solutions to (41)*, which satisfy* (40)*. Then,* $v_1 \equiv v_2$*.*

The existence for the McKean–Vlasov equation

We shall study here the McKean–Vlasov equation (4) under hypotheses (i)–(v), where u is the solution to NFPE (1) with $u_0 \in \mathcal{P} \cap L^\infty$ given by Theorem 1.

As seen earlier, $t \to u(t)$ is narrowly continuous and, therefore, by the superposition principle first applied for the linearized equation (50) and derived afterwards for the NFPE (1), it follows (see [2], [16] and Theorem 7 in [5]) the existence of a probability weak solution X_t to the McKean–Vlasov equation (4) with the law density $u(t)$. More precisely, we have:

There is a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ *with normal filtration* $(\mathcal{F}_t)_{,r>0}$ *and an* $(\mathcal{F}_t)$*-Brownian motion* $(W_t,\ t>0)$ *such that* $X_t,\ t>0$*, is an* $(\mathcal{F}_t)_{,t>0}$ *adapted stochastic process on* $(\Omega, \mathcal{F}, \mathbb{P})$*, which satisfies SDE*

$$dX_t = (D(X_t)b(u(t,X_t)) + (K * u(t,\cdot))X_t)dt + \left(\sqrt{\frac{2\beta(u(t,X_t))}{u(t,X_t)}}\right)dW_t, \forall t>0, \quad (52)$$
$$X(0) = X_0,$$

$$\mathcal{L}_{X_t}(x) = u(t,x),\ \forall t>0,\ u_0(x) = \mathcal{L}_{X_0}(x) = (\mathbb{P} \circ X_0^{-1})(x),\ \forall x \in \mathbb{R}^d. \quad (53)$$

We may view (52)–(53) as a probability representation of solutions u to the NFPE (1). Under hypotheses (i)–(v) we have also from the uniqueness of NFPE (39) and its linearized version (50) the weak uniqueness in law for the solution to the McKean–Vlasov SDE (52). Namely, we have

Theorem 7

Let X_t and $\widetilde{X}_t$ be probability weak solutions to (52) *on the stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbb{P}})$ *such that*

$$\mathcal{L}(X_t) = u(t, \cdot),\ \mathcal{L}(\widetilde{X}_t) = \widetilde{u}(t, \cdot). \tag{54}$$

$$u, \widetilde{u} \in L^1(Q) \cap L^\infty(Q),\ Q = (0, \infty) \times \mathbb{R}^d. \tag{55}$$

Then X_t and $\widetilde{X}_t$ have the same marginal laws, that is,

$$\mathbb{P} \circ X_t^{-1} = \widetilde{\mathbb{P}} \circ \widetilde{X}_t^{-1},\ \forall t > 0. \tag{56}$$

Proof. By Itô's formula, both functions u and $\widetilde{u}$ satisfy (37)–(39) and so, by Theorem 5, $u = \widetilde{u}$. Moreover, both marginal laws $\mathbb{P} \circ X_t^{-1}$ and $\widetilde{\mathbb{P}} \circ \widetilde{X}_t^{-1}$ satisfy the martingale problem with the initial condition u_0 for the linearized Kolmogorov operator

$$v \to \Delta\left(\frac{\beta(u)}{u}\, v\right) - (Db(u) + K * u) \cdot \nabla v$$

and so, by Theorem 4.3 and the transfer of uniqueness Lemma 2.12 in [17] (see also [15] and [5], Sect. 5), it follows that (56) holds. □

We recall that (see, e.g., [5], p. 195) that a *weak probability solution* X_t to (52)–(53) is said to be a *strong solution* if it is a measurable function of the Brownian motion W_t or, equivalently, if it is adapted with respect to the completed natural filtration $(\mathcal{F}_t^{W_t})_{t>0}$ of W_t. It turns out that under appropriate conditions on b, D and K, the weak probability solution X_t is a strong solution to the McKean–Vlasov equation (52)–(53). Namely, we have

Theorem 8

Assume that, besides hypotheses (i)–(v)*, the following conditions hold*

$$t \to \frac{\beta(r)}{r} \in C_b^1[0,\infty),\ D \in C^1(\mathbb{R}^d;\mathbb{R}^d), \tag{57}$$

$$\frac{\partial}{\partial x_i} K \in L^2(B_1^c;\mathbb{R}^d),\ i = 1,...,d. \tag{58}$$

Then, the weak probability solution X_t to (52)–(53) *is the unique strong solution to the McKean–Vlasov equation* (52)–(53)*.*

Proof. By the restricted Yamada–Watanabe theorem (see, e.g., [9], [10]) the conclusion follows if one shows that the stochastic differential equation (52) with u given by Theorem 1 has the path uniqueness property for the weak probability solutions, that is, for every pair of weak solutions $(X, W, (\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t>0})), (\widetilde{X}, W, (\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t>0}))$, with $X(0) = \widetilde{X}(0)$, we have $\sup |X(t) - \widetilde{X}(t)| = 0$, $\mathbb{P}$-a.s. In our case, this means the pathwise uniqueness of weak probability solutions to the stochastic differential equation

$$\begin{aligned} dX_t &= f_1(t, X_t)dt + f_2(t, X_t)dW_t \\ X(0) &= X_0 \in \mathbb{R}^d, \end{aligned} \tag{59}$$

in the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, W_t)$, where

$$\begin{aligned} f_1(t, x) &\equiv D(x)b(u(t, x)) + (K * u(t, \cdot))(x),\ t > 0,\ x \in \mathbb{R}^d, \\ f_2(t, x) &\equiv \sqrt{\frac{2\beta(u(t, x))}{u(t, x)}},\ t > 0,\ x \in \mathbb{R}^d, \end{aligned}$$

and u is the solution to the Fokker–Planck equation (1) given by Theorem 1.

The main difficulty for the proof of the path uniqueness for equation (59) is that the coefficients f_i, $i = 1, 2$, are not Lipschitz with respect to the spatial variable $x \in \mathbb{R}^d$ but are only in H^1. We have indeed for
$D_i = \frac{\partial}{\partial x_i}, \ i = 1, ..., d,$

$$\begin{aligned} |D_i f_2(t,x)| &= \left| \left(\frac{\beta(u)}{u}\right)' \left(\frac{\beta(u)}{u}\right)^{-\frac{1}{2}} D_i u(t,x) \right| \leq C|D_i u(t,x)|, \\ &\qquad \forall (t,x) \in (0,\infty) \times \mathbb{R}^d, \end{aligned}$$

$$\begin{aligned} |D_i(D(x)b(u(t,x)))| &\leq |D|_\infty |b(u)|_\infty + |D|_\infty |b'(u)|_\infty |\nabla u(t,x)| \\ &\leq C|D_i u(t,x)|, \ (t,x) \in (0,\infty) \times \mathbb{R}^d, \end{aligned}$$

$$\begin{aligned} D_i(K * u(t))(x) &= \int_{B_1} K(\bar{x}) D_i u(t, x-\bar{x}) dx + \int_{B_1^c} D_i K(\bar{x}) u(t-\bar{x}) dx \\ &+ \int_{[|\bar{x}|=1]} K(\bar{x}) u(t, x-\bar{x}) d\bar{x}, \ i = 1, ..., d, \ (t,x) \in (0,\infty) \times \mathbb{R}^d. \end{aligned}$$

Taking into account that, as seen by Theorem 1, $u \in L^\infty(0,T;H^1)$, these formulae imply that

$$\begin{aligned}|D_i f_1(t,\cdot)|_2 \leq\ & C|D_i u(t,\cdot)|_2 + \|K\|_{L^1(B_1)}|D_i u(t,\cdot)|_2 \\ & + \|D_i K\|_{L^1(B_1^c)}|u(t,\cdot)_2 + C\|K\|_{L^\infty(B_1)}|u(t,u(t))|_\infty,\ \forall t \in [0,T].\end{aligned}$$

By the above estimations, we see that f_1, f_2 have the H^1-Sobolev regularity with respect to the spatial variable x and so the pathwise uniqueness of the weak solution X to (59) follows by the same argument as that used for the uniqueness of Lagrangian flows generated by nonlinear differential equations in $\mathbb{R}^d$ with coefficients in $L^1(0,T;H^1)$ (see Theorem 2.9 in [7]). $\square$

References

[1] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishers, Leyden, 1976.

[2] Barbu, V., Röckner, M., From Fokker–Planck equations to solutions of distribution dependent SDE, *Ann. Probab.*, 48 (4) (2020), 1902-1920.

[3] Barbu, V., Röckner, M., Solutions for nonlinear Fokker-Planck equations with measures as initial data and McKean–Vlasov equations, *J. Funct. Anal.*, 280 (7) (2021), 1-35.

[4] Barbu, V., Röckner, M., Uniqueness for nonlinear Fokker-Planck equations and for McKean-Vlasov SDEs: The degenerate case, J. Funct. Anal, 285 (4), 109980 (2023).

[5] Barbu, V., Röckner, M., Nonlinear Fokker-Planck flows and their probabilistic counterparts, Lecture Notes in Mathematics, 2353 (2014), Springer.

[6] Barbu, V., Röckner, M., Zhang, D., Uniqueness of distributional solutions to the 2D vorticity Navier–Stokes equation and its associated Markov process, *J. European Math. Soc.* (to appear).

[7] Crippa, G., De Lellis, C., Estimates and regularity results for the DiPerna–Lions flow, *J. Reine Angew. Math.*, 616 (2008), 15-46.

[8] Flandoli, F., Leimbach, M., Olivera, C., Uniform convergence of proliferating particles to the FKPP equation, *J. Math. Anal. Appl.*, 473 (1) (2019), 27-52.

[9] Grube, S., Strong solutions to McKean–Vlasov SDEs with coefficients of Nemitskii-type, *Electronic J. Comm. Probab.*, 28 (2023), 1-13.

[10] Grube, S., Strong solutions to McKean–Vlasov SDEs with coefficients of Nemitskii type, *J. Evol. Equations*, 2 (2024), 1-14.

[11] Keller, J., Segal, L.A., Model for chemotaxis, *J. Theoret. Biology*, 30 (1971), 225-234.

[12] Kōmura, Y., Differentiability of nonlinear semigroups, *J. Math. Soc. Japan*, vol. 21, 3 (1969), 375-402.

[13] Olivera, C., Richard, A., Tomasevic, M., Quantitative particles approximation of nonlinear Fokker-Planck equations with singular kernel, *Ann. Scuola Sup. Pisa, Classe Scienze*, vol. XXV (2023), 691-749.

[14] Painter, J.K., Mathematical models for chemotaxis and their applications in self-organization phenomena, *J. Theoretical Biology*, 481 (2019), 162-182.

[15] Ren, P., Röckner, M., Wang, F.Y., Linearization of nonlinear Fokker–Planck equations and applications, *J.Diff.Eqns.*, 322 (2022), 1-37.

[16] Röckner, M., Zhang, X., Well posedness of distribution dependent SDEs with singular drifts, *Bernoulli*, 27 (2021), 1131-1158.

[17] Trevisan, D., Well-posedness of multi-dimensional diffusion processes with weakly differentiable coefficients, *Electron. J. Probab.*, 21 (22) (2016), 1-41.