KAM-rigidity for parabolic affine abelian actions on the torus

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Abstract: Two famous instances of local rigidity for $\mathbb{Z}^2$-actions are the classical KAM rigidity of Diophantine toral translations and smooth rigidity of hyperbolic or partially hyperbolic higher rank actions proved by Damjanovic and Katok. To complete the study of local rigidity of affine $\mathbb{Z}^2$-actions on the torus, we address the case of parabolic affine actions.

Consider an affine $\mathbb{Z}^2$-action (a, b) on $\mathbb{T}^d$ generated by two commuting parabolic affine maps of the form $a(x) = A(x)+\alpha$, $b(x) = B(x)+\beta$, where $A, B \in SL(d, \mathbb{Z})$.

We say that the action (a, b) is *KAM-rigid under volume-preserving perturbations* if there exists $\sigma \in \mathbb{N}$, $r_0 \geq \sigma$ and $\varepsilon > 0$ satisfying the following. If $r \geq r_0$ and $(F, G) = (a + f, b + g)$ is a smooth λ-preserving $\mathbb{Z}^2$-action such that

$$\|f\|_r \leq \varepsilon, \quad \|g\|_r \leq \varepsilon, \quad \widehat{f} := \int_{\mathbb{T}^d} f d\lambda = 0, \quad \widehat{g} := \int_{\mathbb{T}^d} g d\lambda = 0,$$

then there exists $H = \mathrm{Id} + h \in \mathrm{Diff}^\infty_\lambda(\mathbb{T}^d)$ such that $\|h\|_{r-\sigma} \leq \varepsilon$ and

$$H \circ (a + f) \circ H^{-1} = a, \quad H \circ (b + g) \circ H^{-1} = b.$$

Let $\mathcal{T}(A, B)$ denote the set of possible translation parts (α, β) in the affine actions with linear part (A, B), that is $\mathcal{T}(A, B) := \{\alpha, \beta \in \mathbb{R}^d \mid (A - \mathrm{Id})\beta = (B - \mathrm{Id})\alpha\}$.

We present the following dichotomy for a commuting pair (A, B) of parabolic matrices, where A is step-2 (i.e., $(A - Id)^2 = 0$):

- (i) either for any choice of $(\alpha, \beta) \in \mathcal{T}(A, B)$, the affine action (a, b) has a rank one factor that is different from a nonzero translation, in which case the action is not KAM-rigid,
- (ii) or for almost every choice of $(\alpha, \beta) \in \mathcal{T}(A, B)$ the action (a, b) is ergodic and KAM-rigid under volume preserving perturbations.

This is the result of a joint work with D. Damjanovic and B. Fayad.