Bimodule connections for line bundles over irreducible quantum flag manifolds.

Based on a joint work with Réamonn Ó Buachalla

Quantum and Super day in Bologna 2022

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September 8, 2022

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- In the noncommutative setting it is most natural to consider **bimodule connections**, which is to say, left (right) connections, together with a compatibility for the right (left) action, described in terms of an associated bimodule map.
- Not many examples of bimodule connections were known in literature. In the world of quantum groups Beggs and Majid produced bimodule connections for the line bundles over the Podle's sphere while Matassa did the same for its Heckenberger and Kolb calculus.

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- Now associated modules are naturally bimodules, and so, it is natural to ask when a principal connection induces a bimodule connection on an associated module.
- We show how to define a bimodule connection on homogeneous line bundles on the **irreducible quantum flag manifolds** starting from from a **principal connection**.

Definition

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank l. Fix a Cartan subalgebra $\mathfrak{h}$ and choose a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_l\}$. Let (a_{ij}) be the Cartan matrix of $\mathfrak{g}$ associated to $\mathfrak{h}^*$.
Let $q \in \mathbb{R}$ such that $q \neq -1, 0, 1$. The **Drinfeld–Jimbo quantised enveloping algebra** *$U_q(\mathfrak{g})$ is the associative algebra generated by the elements E_i, F_i, K_i, and K_i^{-1}, for $i = 1, \dots, l$, subject to the relations*

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}}F_jK_i, \quad K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations.

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As matter of fact, $U_q(\mathfrak{g})$ has an the structure of an Hopf algebra.

Definition

For any $U_q(\mathfrak{g})$-module V, a vector $v \in V$ is called a **weight vector** *of* **weight** $\mathrm{wt}(v) \in \mathcal{P}$ *if*

$$K_i \triangleright v = q^{(\alpha_i, \mathrm{wt}(v))} v, \qquad \textit{for all } i = 1, \dots, r.$$

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Let $\mathcal{P}^+$ be the $\mathbb{Z}_{\geq 0}-$span of the fundamental weights of $U_q(\mathfrak{g})$. For each $\lambda \in \mathcal{P}^+$ there exists an irreducible finite-dimensional $U_q(\mathfrak{g})$-module V_λ, uniquely defined by the existence of a weight vector $v_{\mathrm{hw}} \in V_\lambda$ of weight λ, which we call a **highest weight vector**, satisfying $E_i \triangleright v_{\mathrm{hw}} = 0$, for all $i = 1, \ldots, l$.

Definition

Let $V \in {}_{U_q(\mathfrak{g})}\mathrm{mod}$, For $v \in V$ and $f \in V^*$, consider the function

$$c^V_{f,v} : U_q(\mathfrak{g}) \to \mathbb{C}$$
$$c^V_{f,v}(X) := f\big(X(v)\big).$$

The **coordinate ring** *of V is the subspace*

$$C(V) := span_{\mathbb{C}}\left\{c^V_{f,v} \,|\, v \in V,\ f \in V^*\right\} \subseteq U_q(\mathfrak{g})^*.$$

$C(V)$ is contained in $U_q(\mathfrak{g})^\circ$, the **Hopf dual** *of $U_q(\mathfrak{g})$, a Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ is given by*

$$\mathcal{O}_q(G) := \bigoplus_{\lambda \in \mathcal{P}^+} C(V_\lambda).$$

We call $\mathcal{O}_q(G)$ the **quantum coordinate algebra of** *G, where G is the compact, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra.*

Definition

A **differential calculus** $(\Omega^\bullet \simeq \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \Omega^k, \mathrm{d})$ *is a differential graded algebra (dg-algebra) which is generated in degree* 0 *as a dg-algebra, that is to say, it is generated as an algebra by the elements* $a, \mathrm{d}b$*, for* $a, b \in \Omega^0$*. For a given algebra* B*, a differential calculus* **over** B *is a differential calculus such that* $B = \Omega^0$*.*

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Definition

A first-order differential calculus*(fodc) over an algebra* B *is a pair* (Ω^1, d)*, where* Ω^1 *is a* B*-bimodule and* $\mathrm{d} : B \to \Omega^1$ *is a derivation such that* Ω^1 *is generated as a left* B*-module by those elements of the form* $\mathrm{d}b$*, for* $b \in B$*. We call* d **the exterior derivative** *of the fodc.*

A first order differential calculus always admits an extension as differential calculus.

A Complex structure abstracts the properties of the de Rham complex of a classical complex manifold.

Definition

A **complex structure** $\Omega^{(\bullet,\bullet)}$ *for a differential* $*$*-calculus* $(\Omega^\bullet, \mathrm{d})$ *is a choice of* $\mathbb{Z}^2_{\geq 0}$*-algebra grading* $\bigoplus_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \Omega^{(a,b)}$ *for* $\Omega^\bullet$ *such that*

$$1.\ \Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}, \quad 2.\ \left(\Omega^{(a,b)}\right)^* = \Omega^{(b,a)}, \quad 3.\ \mathrm{d}\Omega^{(a,b)} \subseteq \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)},$$

for all $k \in \mathbb{Z}_{\geq 0}$*, and* $(a,b) \in \mathbb{Z}^2_{\geq 0}$*.*

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for all $k \in \mathbb{Z}_{\geq 0}$*, and* $(a,b) \in \mathbb{Z}^2_{\geq 0}$*.*

We call an element of $\Omega^{(a,b)}$ an (a,b)-form, and denote

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

∂ and $\overline{\partial}$ are $*$-maps and satisfy the graded Leibniz rule. A *covariant* complex structure is an integrable covariant almost complex structure. We note that for a covariant complex structure, the maps ∂ and $\overline{\partial}$ are left A-comodule maps.

Definition

We say that a differential calculus $(\Omega^\bullet, \mathrm{d})$ *is* **left $U_q(\mathfrak{g})$-covariant** *if* $\Omega^\bullet$ *admits a* $U_q(\mathfrak{g})$*-left module structure and the differential* $\mathrm{d} : \Omega^\bullet \to \Omega^\bullet$ *is a left* $U_q(\mathfrak{g})$*-module map. (We define simlarly right covariance and bicovariance). We say that a complex structure for* $\Omega^\bullet$ *is covariant if the* $\mathbb{Z}^2_{\geq 0}$*-decomposition is a decomposition in* ${}^{U_q(\mathfrak{g})}\mathrm{Mod}$.

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PROBLEM:

There exist no bicovariant differential calculus on $\mathcal{O}_q(G)$ of classical dimension!

However this problem can be avoided if we decide to study a particular class of homogeneous spaces of $\mathcal{O}_q(G)$.

Let A be an Hopf algebra consider a left coideal subalgebra $B \subseteq A$ such that $B^+A = AB^+$.
We have a $\pi_B(A)$-coaction

$$\Delta_{R,\pi_B} := (\mathrm{id} \otimes \pi_B) \circ \Delta, \qquad \pi_B : A \to A/B^+A$$

We have $A^{\mathrm{co}(A/B^+A)} = B$.

Definition

If A is faithfully flat as a right B-module, we call B a **quantum homogeneous A-space**.

Definition

For S a subset of simple roots, consider the Hopf subalgebra of $U_q(\mathfrak{g})$ given by

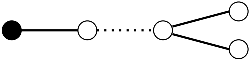
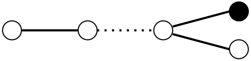
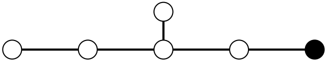
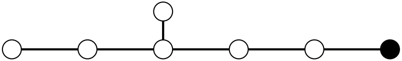
$$U_q(\mathfrak{l}_S) := \left\langle K_i, E_j, F_j \,|\, i = 1, \ldots, l; j \in S \right\rangle.$$

Consider now the coideal subalgebra of $U_q(\mathfrak{l}_S)$-invariants

$$\mathcal{O}_q\big(G/L_S\big) := {}^{U_q(\mathfrak{l}_S)}\mathcal{O}_q(G),$$

with respect to the natural left $U_q(\mathfrak{g})$-module structure on $\mathcal{O}_q(G)$. Cosemisimplicity of the Hopf dual of $U_q(\mathfrak{l}_S)$ implies that $\mathcal{O}_q(G/L_S)$ is a quantum homogeneous space. We call it the **quantum flag manifold associated to** *S.*

Irreducible quantum flag manifolds

A_n	$\mathcal{O}_q(\mathrm{SU}_{n+1})$		$\mathcal{O}_q(\mathrm{Gr}_{n+1,m})$
B_n	$\mathcal{O}_q(\mathrm{Spin}_{2n+1})$		$\mathcal{O}_q(\mathbf{Q}_{2n+1})$
C_n	$\mathcal{O}_q(\mathrm{Sp}_n)$		$\mathcal{O}_q(\mathbf{L}_n)$
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{Q}_{2n})$
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{S}_n)$
E_6	$\mathcal{O}_q(E_6)$		$\mathcal{O}_q(\mathbb{OP}^2)$
E_7	$\mathcal{O}_q(E_7)$		$\mathcal{O}_q(\mathbf{F})$

Theorem (Heckenberger, Kolb '06)

For each quantum flag manifold $\mathcal{O}_q(G/L_S)$ of Hermitian symmetric type, there exists a unique right covariant differential calculus $\Omega_q^\bullet(G/L_S)$ of classical dimension.

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For each quantum flag manifold $\mathcal{O}_q(G/L_S)$ of Hermitian symmetric type, there exists a unique right covariant differential calculus $\Omega_q^\bullet(G/L_S)$ of classical dimension.

- This differential calculus exhibits a complex stuctures, Hodge map and even noncommutative Kähler structure.
- We call $\Omega_q^{(1,0)}(G/L_S)$ and $\Omega_q^{(0,1)}(G/L_S)$ the **holomorphic** and **antiholomorphic** calculi respectively.

Definition

For Ω^1 a fodc over an algebra B, and $\mathcal{F}$ a left B-module, a **left connection** *on $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ satisfying*

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla f, \qquad \textit{for all } b \in B, f \in \mathcal{F}.$$

A left bimodule connection on $\mathcal{F}$ is a pair (∇, σ) where ∇ is a left connection and $\sigma : \mathcal{F} \otimes_B \Omega^1(B) \to \Omega^1(B) \otimes_B \mathcal{F}$ is a bimodule map satisfying

$$\sigma(f \otimes_B db) = \nabla(fb) - \nabla(f)b. \tag{1}$$

Classically the algebra $\mathfrak{l}_S$ is reductive, and hence decomposes into a direct sum $\mathfrak{l}^{\mathrm{s}}_S \oplus \mathfrak{u}_1$, comprised of a semisimple part and a commutative part, respectively. We are thus motivated to consider the Hopf subalgebra

$$U_q(\mathfrak{l}^{\mathrm{s}}_S) := \big\langle K_i, E_i, F_i \,|\, i \in S \big\rangle \subseteq U_q(\mathfrak{l}_S).$$

And the coideal subalgebra of $U_q(\mathfrak{l}^{\mathrm{s}}_S)$-invariants

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Definition

We call $\mathcal{O}_q\big(G/L_S^{\mathrm{s}}\big)$ the quantum homogeneous Poisson space associated to S.

Definition

A **line module** *over B is be an invertible B-bimodule $\mathcal{E}$: there exists another B-bimodule $\mathcal{E}^\vee$ such that*

$$\mathcal{E} \otimes_B \mathcal{E}^\vee \simeq \mathcal{E}^\vee \otimes_B \mathcal{E} \simeq B. \tag{2}$$

A **relative line module** *over a quantum homogeneous A-space B is a B-sub-bimodule, left A-comodule, $\mathcal{E} \subseteq A$ that is also a line module over B, and for which the isomorphisms above are left A-comodule maps.*

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Proposition

$\mathcal{O}_q(G/L_S^{\mathrm{s}})$ is decomposed into simple subobjects as a direct sum of relative line modules

$$\mathcal{O}_q(G/L_S^{\mathrm{s}}) \simeq \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i.$$

This gives $\mathcal{O}_q(G/L_S^{\mathrm{s}})$ the structure of a strongly $\mathbb{Z}$-graded algebra. Moreover, every relative line module over $\mathcal{O}_q(G/L_S)$ is isomorphic to $\mathcal{E}_i$, for some $i \in \mathbb{Z}$.

Theorem (F. Díaz García, A. Krutov, R. Ó Buachalla, P. Somberg, K. R. Strung)

For each relative line module $\mathcal{E}_k$ *there exists a unique left* $\mathcal{O}_q(G)$*-covariant connection* $\overline{\partial}_{\mathcal{E}_k} : \mathcal{E}_k \to \Omega_q^{(0,1)}(G/L_S) \otimes_{\mathcal{O}_q(G/L_S)} \mathcal{E}_k$.

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For each line bundle $\mathcal{E}_k$, we gave a construction of this unique connections, presented as associated to a principal connection.

Definition (Brzeziński–Majid)

Let H be a Hopf algebra. A **quantum principal** H-**bundle** *is a pair* $(P, \Omega^1(P))$*, consisting of a right H-comodule algebra* (P, Δ_R) *and a right-H-covariant calculus* $\Omega^1(P)$*, such that:*

1. *P is a Hopf–Galois extension of* $B = P^{\mathrm{co}(H)}$.
2. *If* $N \subseteq \Omega^1_u(P)$ *is the sub-bimodule of the universal calculus corresponding to* $\Omega^1(P)$*, we have* $\mathrm{ver}(N) = P \otimes I$*, for some* Ad*-sub-comodule right ideal*

$$I \subseteq H^+ := \ker(\varepsilon : H \to \mathbb{C}).$$

An exact short sequence is then given by

$$0 \longrightarrow P\Omega^1(B)P \xrightarrow{\iota} \Omega^1(P) \xrightarrow{\mathrm{ver}} P \otimes \Lambda^1(H) \longrightarrow 0$$

Where $\mathrm{ver} := \mathrm{can} \circ \mathrm{proj}_B$ and $\mathrm{Ad} : H \to H \otimes H$ is defined by $\mathrm{Ad}(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$.

Definition

A **principal connection** *for a quantum principal H-bundle $(P, \Omega^1(P))$ is a left P-module, right H-comodule, projection $\Pi : \Omega^1(P) \to \Omega^1(P)$ satisfying*

$$\ker(\Pi) = P\Omega^1(B)P.$$

A principal connection Π is called **strong** *if $(\mathrm{id} - \Pi)(\mathrm{d}P) \subseteq \Omega^1(B)P$.*

For any left B-submodule right H-subcomodule $\mathcal{E} \subseteq P$, we can use a strong principal connection to define a connection $\nabla : \mathcal{E} \to \Omega^1(B) \otimes_B \mathcal{E}$.

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- The multiplication map gives an isomorphism

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- The multiplication map gives an isomorphism

$$j : \Omega^1(B) \otimes_B \mathcal{E} \hookrightarrow \Omega^1(B)\mathcal{E},$$

- A strong principal connection Π defines a connection ∇ on $\mathcal{E}$ by

$$\nabla := j^{-1} \circ (\mathrm{id} - \Pi) \circ \mathrm{d} : \mathcal{E} \to \Omega^1(B) \otimes_B \mathcal{E}.$$

Proposition (A.C.,Fredy Díaz García and Réamonn Ó Buachalla)

The pair

$$\left(\mathcal{O}_q(G/L_S^{\mathrm{s}}), \Omega_q^{(0,1)}(G/L_S^{\mathrm{s}})\right)$$

is a quantum principal bundle.

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The zero map on $\Omega_q^{(0,1)}(G/L_S^{\mathrm{s}})_{\mathrm{hor}}$ is a left $\mathcal{O}_q(G)$-covariant strong principal connection.

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The zero map on $\Omega_q^{(0,1)}(G/L_S^{\mathrm{s}})_{\mathrm{hor}}$ is a left $\mathcal{O}_q(G)$-covariant strong principal connection.

Thus we have a principal connection presentation of the unique connection on $\mathcal{E}_k$.

$$\overline{\partial}_{\mathcal{E}_k} : \mathcal{E}_k \to \Omega^{(0,1)} \otimes_{\mathcal{O}_q(G/L_S)} \mathcal{E}_k, \qquad e \mapsto j^{-1} \circ \overline{\partial}(e),$$

Theorem (A.C. and Réamonn Ó Buachalla)

For each line module $\mathcal{E}_k$, the connection ∇ is a bimodule connection. Moreover, the associated bimodule map is a left $\mathcal{O}_q(G)$-comodule isomorphism.

We can give two complementary presentations of the bimodule map associated to the connection $\overline{\partial}_{\mathcal{E}_k}$

- In terms of generalised quantum determinants, taking also advantage of the FRT presentation of $\mathcal{O}_q(G)$.
- In terms of Takeuchi's categorical equivalence for relative Hopf modules.

- We have a dual pair of the irreducible $U_q(\mathfrak{g})$-modules,

$$V_k := U_q(\mathfrak{g}) z_N^k, \qquad\qquad {}^\vee V_k := U_q(\mathfrak{g}) \overline{z}_N^k.$$

- Since both $V_k \otimes V_k^\vee$ and ${}^\vee V_k \otimes V_k$ contain a copy of the trivial module, we must have a family of elements

$$\{f_i,\, f_i'\}_{i=1}^N \subseteq V_k, \qquad\qquad \{v_i,\, v_i'\}_{i=1}^N \subseteq {}^\vee V_k,$$

satisfying

$$\sum_{i=1}^N f_i v_i = 1, \qquad\qquad \sum_{i=1}^N v_i' f_i' = 1.$$

- We have

$$\overline{\partial} f_i = \overline{\partial} f_i' = 0, \qquad\qquad \text{for all } i = 1, \ldots, N,$$

Proposition

For $k \in \mathbb{Z}_{>0}$, the bimodule map associated to the connection $\overline{\partial}_{\mathcal{E}_k}$, satisfies

$$\sigma(e \otimes \overline{\partial} b) = \sum_{i=1}^{N} \overline{\partial}(ebv_i') \otimes f_i' - \sum_{i=1}^{N} \overline{\partial}(ev_i') \otimes f_i' b,$$

moreover, the inverse map σ^{-1} satisfies

$$\sigma^{-1}(e \otimes \overline{\partial} b) = \sum_{i=1}^{N} f_i \otimes \overline{\partial}(v_i be) - \sum_{i=1}^{N} bf_i \otimes \overline{\partial}(v_i e),$$

A similar result holds for the bimodule map associated to $\overline{\partial}_{\mathcal{E}_{-k}}$.

Example (The Podleś sphere $\mathcal{O}_q(S^2)$.)

Denoting by u^i_j, for $i,j = 1,2$, the standard generators of $\mathcal{O}_q(\mathrm{SU}_2)$ the quantum determinant relations

$$u_{11}u_{22} - qu_{21}u_{12} = 1$$

This means that for $e \in \mathcal{E}_1$, we have

$$\begin{aligned}\sigma_1(e \otimes \overline{\partial} b) =& \overline{\partial}(ebu_{22}) \otimes u_{11} - \overline{\partial}(eu_{22}) \otimes u_{11}b - \\ & q^{-1}(\overline{\partial}(ebu_{12}) \otimes u_{21} - \overline{\partial}(eu_{12}) \otimes u_{21}b).\end{aligned}$$

Example (quantum Grassmannian $\mathcal{O}_q(\mathrm{Gr}_{n,m})$)

- *The* **quantum minor** $[I|J] \in \mathcal{O}_q(\mathrm{SU}_n)$ *is given by*

$$[I|J] := \sum_{\sigma \in S_p} (-q)^{\ell(\sigma)} u_{\sigma(i_1)j_1} \cdots u_{\sigma(i_p)j_p} = \sum_{\sigma \in S_p} (-q)^{\ell(\sigma)} u_{i_1\sigma(j_1)} \cdots u_{i_p\sigma(j_p)}.$$

- *The quantum Poisson homogeneous space* $\mathcal{O}_q(S^{n,m})$ *of the quantum Grassmannian* $\mathcal{O}_q(\mathrm{Gr}_{n,m})$ *is generated by the quantum minors*

$$z_A := [A|M^\perp], \qquad \overline{z}_B := [B|M], \qquad \textit{for } |A| = n-m,\ |B| = m,$$

- *We have the standard identity*

$$1 = \sum_{|A|=m} q^{\ell_\perp(M,A)} \overline{z}_{A^\perp} z_A,$$

which is a generalized determinant relation.

Example

- *For* $e \in \mathcal{E}_1$, *we see that*

$$\sigma_1(e \otimes \overline{\partial} b) = \sum_{|A|=n-r} q^{\ell_\perp(M,A)} \overline{\partial}(eb\overline{z}_{A^\perp}) \otimes z_A - \sum_{|A|=n-r} q^{\ell_\perp(M,A)} \overline{\partial}(e\overline{z}_{A^\perp}) \otimes z_A b.$$

- *We can also produce explicit formula for the connections* $\overline{\partial}_{\mathcal{E}_k}$*: For for* $e \in \mathcal{E}_1$, *it holds that*

$$\overline{\partial}_{\mathcal{E}_{-1}}(e) = \sum_{|A|=n-r} q^{\ell_\perp(M,A)} \overline{\partial}(e\overline{z}_{A^\perp}) \otimes z_A.$$

- Consider the category of ${}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0,$ of relative Hopf modules $\mathcal{F}$ such that $\mathcal{F}B^+ = B^+\mathcal{F}$.
- The calculi $\Omega_q^{(0,1)}(G/L_S)$ and $\Omega_q^{(1,0)}(G/L_S)$ live in this category.

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- We define a functor

$$\Phi : {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_{\mathcal{O}_q(G/L_S)} \to {}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_{\mathcal{O}_q(G/L_S)},$$

by setting $\Phi(\mathcal{F}) := \mathcal{F}/B^+\mathcal{F}$, with the left $\mathcal{O}_q(L_S)$-comodule structure of $\Phi(\mathcal{F})$ given by

$$\Delta_L[f] := \pi_S(f_{(-1)}) \otimes [f_{(0)}]$$

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- In the other direction

$$\Psi : {}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_{\mathcal{O}_q(G/L_S)} \to {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_{\mathcal{O}_q(G/L_S)},$$

we have the cotensor product $\Psi(V) := \mathcal{O}_q(G)\Box_{\mathcal{O}_q(L_S)}V$.

- Since $\mathcal{O}_q(G)$ is faithfully flat as a right $\mathcal{O}_q(G/L_S)$-module, it follows from Takeuchi's equivalence for quantum homogeneous spaces that Φ induces an equivalence of monoidal categories.
- Since any σ is clearly a morphism in ${}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$, , we can completely describe it by identifying its image under Φ.

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Proposition

For any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, the map

$$\Phi(\sigma_k) : \Phi(\mathcal{E}_k) \otimes \Phi(\Omega_q^{(0,1)}(G/L_S)) \to \Phi(\Omega_q^{(0,1)}(G/L_S)) \otimes \Phi(\mathcal{E}_k),$$

for any line module $\mathcal{E}_k$, with $k \in \mathbb{Z}$, satisfies

$$\Phi(\sigma_k)([e] \otimes [\omega]) = \theta^k [\omega] \otimes [e],$$

for a nonzero scalar $\theta \in \mathbb{C}$.

Example (the Podleś sphere $\mathcal{O}_q(S^2)$)

- *For $e \in \mathcal{E}_1$, the element $\Phi(\sigma_1)[e \otimes \overline{\partial} b]$ is equal to*

$$\Phi(\sigma_1)[e \otimes \overline{\partial} b] = [\overline{\partial}(ebu_{11}) \otimes u_{22}] - [\overline{\partial}(eu_{11}) \otimes u_{22} b] - q([\overline{\partial}(ebu_{21}) \otimes u_{12}] - q[\overline{\partial}(eu_{21}) \otimes u_{12} b]).$$

- *Choosing $e = u_{22}$ and $b = u_{12}u_{22}$, we get*

$$\Phi(\sigma_1)[u_{22} \otimes \overline{\partial} u_{12} u_{22}] = q^{-1}[\overline{\partial}(u_{12}u_{22}) \otimes u_{22}].$$

- *From which it follows that $\theta = q^{-1}$.*

Finally we can pass again to the "global" picture with the following.

Proposition

It holds that

$$\sigma_k(e \otimes \overline{\partial} b) = \theta^k e_{(-1)} b_{(1)} S(e_{(-1)}) S(b_{(2)}) \overline{\partial} b_{(3)} \otimes e_{(0)},$$

for any $b \in \mathcal{O}_q(G/L_S)$, and $e \in \mathcal{E}_k$.

Proof.

We have that

$$\sigma(e \otimes \overline{\partial} b) = \mathrm{U}^{-1} \circ (\mathrm{id} \otimes \Phi(\sigma)) \circ \mathrm{U}(e \otimes \overline{\partial} b).$$

Recalling the explicit presentation of the inverse of U

$$\begin{aligned} \theta^k \mathrm{U}^{-1}(e_{(-1)} b_{(1)} \otimes [\overline{\partial} b_{(2)} \otimes e_{(0)}]) &= \theta^k e_{(-1)} b_{(1)} S(b_{(2)} e_{(-1)}) \overline{\partial} b_{(3)} \otimes e_{(0)} \\ &= \theta^k e_{(-1)} b_{(1)} S(e_{(-1)}) S(b_{(2)}) \overline{\partial} b_{(3)} \otimes e_{(0)}, \end{aligned}$$

giving us the claimed identity. □

Example

For the commutative case, we clearly have that $\theta = 1$. Our formula reduces to

$$\begin{aligned}\sigma_k(e \otimes \overline{\partial} b) &= e_{(-1)} b_{(1)} S(e_{(-1)}) S(b_{(2)}) \overline{\partial} b_{(3)} \otimes e_{(0)} \\ &= e_{(-2)} S(e_{(-1)}) b_{(1)} S(b_{(2)}) \overline{\partial} b_{(3)} \otimes e_{(0)} \\ &= \overline{\partial} b \otimes e,\end{aligned}$$

giving the flip map, as it should.

Thank you!